Homothetic Motions at $E^4_{\alpha\beta}$

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Abstract

In this paper, a matrix corresponding to Hamilton operators is defined for generalized quaternions is determined a Hamilton motion in four-dimensional space $E^4_{\alpha\beta}$. It is shown that this is a homothetic motion. Also, it is found that the Hamilton motion defined by a regular curve of order $r$ has only one acceleration center of order $(r-1)$ at every instant $t$.

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1 Introduction

To investigate the geometry of the motion of a line or a point in the motion of space is important in the study of space kinematics or spatial mechanisms or in physics. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of mechanical products or in the design of robotic motions. Hacısıalihoğlu[3] showed some properties of 1-parameter homothetic motion in Euclidean space $E^n$. In addition, he found that this motion is regular and has one pole point at every $t$-instant. After him, Yaylı[7] gave homothetic motions with aid of the Hamilton operators in four-dimensional Euclidean space $E^4$. Subsequently, Kula and Yaylı[5] expressed Hamilton motions by means of Hamilton operators in semi-Euclidean space $E^4_2$ and showed that this motions, are a homothetic motion. Also, this subject is investigated in algebra[2]. Recently, we studied the generalized quaternions, and presented some of their algebraic properties[4]. Furthermore, we give some algebraic properties of Hamilton operators of generalized quaternion. In [4], generalized quaternions have expressed in terms
of $4 \times 4$ matrices by means of these operators. In this paper, first, we define a motion by using these matrices, and show that this motion is a homothetic motion in four-dimensional space $E^4_{\alpha\beta}$. We find that the homothetic motion has only one pole point at every instant $t$, and prove that this motion has only one acceleration center of high order at every instant $t$.

2 Preliminaries

Definition 1. A generalized quaternion $q$ is defined as

$$q = a + a_1 i + a_2 j + a_3 k$$

where $a, a_1, a_2$ and $a_3$ are real numbers and 1, $i, j, k$ of $q$ may be interpreted as the four basic vectors of cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta$$

$$ij = k = -ji, \quad jk = \beta i = -kj$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$  

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$. So, a generalized quaternion $q$ is a sum of a scalar and a vector, called scalar part, $S_q = a$, and vector part $V_q = a_1 i + a_2 j + a_3 k \in \mathbb{R}^3_{\alpha\beta}$. Therefore, $H_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis $\mathbb{R}$ and a 3-dimensional real linear space $\mathbb{R}^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = \mathbb{R} \oplus \mathbb{R}^3_{\alpha\beta}$. It is clear, if $\alpha = \beta = 1$ then $H_{11} = H$ (real quaternions), and if $\alpha = 1, \beta = -1$ then $H_{1-1} = H'(split\ quaternions)$ [4].

Definition 2. We define a generalized inner product in $\mathbb{R}^4$,

$$\langle u, v \rangle = u_1 v_1 + \alpha u_2 v_2 + \beta u_3 v_3 + \alpha\beta u_4 v_4$$

where $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$. We put $E^4_{\alpha\beta} = (\mathbb{R}^4, \langle \cdot, \cdot \rangle)$. So, we identity $H_{\alpha\beta}$ with the 4-dimensional space $E^4_{\alpha\beta}$.

Definition 3. A matrix $A$ is called a quasi-orthogonal matrix if $A^T \varepsilon A = \varepsilon$

and det $A = 1$, where

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix} \text{ and } \alpha, \beta \in \mathbb{R} \ [4].$$
3 Homothetic motions in $E^4_{\alpha\beta}$

The 1-parameter homothetic motions of a body in four-dimensional space $E^4_{\alpha\beta}$ is generated by transformation

$$
\begin{bmatrix}
Y \\
1
\end{bmatrix} =
\begin{bmatrix}
hA & C \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
X \\
1
\end{bmatrix}
$$

where $A$ is a quasi-orthogonal matrix. The matrix $B = hA$ is called a homothetic matrix and $Y, X$ and $C$ are $n \times 1$ real matrices. The homothetic scalar $h$ and the elements of $A$ and $C$ are continuously differentiable functions of a real parameter $t$. $Y$ and $X$ correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space $R$ and the fixed space $R_0$, respectively. At the initial time $t = t_0$, we consider the coordinate systems of $R$ and $R_0$ as coincident. To avoid the case of affine transformation we assume that

$$h = h(t) \neq \text{cons.}, \ h(t) \neq 0.$$

and to avoid the case of a pure translation or a pure rotation, we also assume that

$$\frac{d}{dt}(hA) \neq 0, \ \frac{d}{dt}(C) \neq 0.$$

4 Hamilton motions in $E^4_{\alpha\beta}$

Let $q = a_1 + a_1 i + a_2 j + a_3 k$ be a generalized quaternion, and let $h_q : H_{\alpha\beta} \to H_{\alpha\beta}$, $h_q(x) = qx$. The matrix of $h_q$ relative to the natural basis \{1, i, j, k\} for $H_{\alpha\beta}$ is

$$H(q) =
\begin{bmatrix}
a & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\
a_1 & a & -\beta a_3 & \beta a_2 \\
a_2 & \alpha a_3 & a & -\alpha a_1 \\
a_3 & -a_2 & a_1 & a
\end{bmatrix}
$$

(see [4]).

Let us consider the following curve:

$$a : \ I \subset \mathbb{R} \to E^4_{\alpha\beta}
$$

$$a(t) = [a(t), a_1(t), a_2(t), a_3(t)], \ \forall t \in I$$
we suppose that the unit velocity curve $\mathbf{a}(t)$ is differentiable regular curve of order $r$. The operator $B$ called the generalized Hamiltonian operator, corresponding to $\mathbf{a}(t)$ is defined by the following matrix;

$$B = H[\mathbf{a}(t)] = \begin{bmatrix}
a_1(t) & -\alpha a_1(t) & -\beta a_2(t) & -\alpha \beta a_3(t) \\
a_1(t) & a_1(t) & -\beta a_3(t) & \beta a_2(t) \\
a_2(t) & \alpha a_1(t) & a_2(t) & -\alpha a_1(t) \\
a_3(t) & -a_2(t) & a_1(t) & a_1(t) \\
a_2(t) & a_2(t) & -a_1(t) & a_2(t) \\
a_3(t) & a_3(t) & a_1(t) & a_3(t)
\end{bmatrix}. \quad (2)$$

\begin{definition}
The 1-parameter Hamilton motions of a body in $E^4_{\alpha\beta}$ are generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

or equivalently

$$Y = BX + C. \quad (3)$$
\end{definition}

Here $B = H[\mathbf{a}(t)]$ and $Y, X$ and $C$ are $n \times 1$ real matrices, $A$ and $C$ are continuously differentiable functions of a real parameter $t$; $Y$ and $X$ correspond to the position vectors of the same point $P$.

\begin{theorem}
The Hamilton motion determined by equation (3) is a homothetic motion in $E^4_{\alpha\beta}$.
\end{theorem}

\begin{proof}
We suppose that length of $\mathbf{a}(t)$ is not zero, so the matrix $B$ can be represented as

$$B = h \begin{bmatrix}
a(1)(t) & -\alpha a_1(t) & -\beta a_2(t) & -\alpha \beta a_3(t) \\
a_1(t) & a_1(t) & -\beta a_3(t) & \beta a_2(t) \\
a_2(t) & \alpha a_1(t) & a_2(t) & -\alpha a_1(t) \\
a_3(t) & -a_2(t) & a_1(t) & a_1(t) \\
a_2(t) & a_2(t) & -a_1(t) & a_2(t) \\
a_3(t) & a_3(t) & a_1(t) & a_3(t)
\end{bmatrix} = hA \quad (4)$$

where $h : I \subset \mathbb{R} \to \mathbb{R},$

$$t \to h(t) = ||\mathbf{a}(t)|| = \sqrt{a^2(t) + \alpha a_1^2(t) + \beta a_2^2(t) + \alpha \beta a_3^2(t)}$$

so, we find $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, thus $B$ is a homothetic matrix and equation (3) determines a homothetic motion.
\end{proof}

\begin{special_cases}
(i) For the case $\alpha = \beta = 1$, $A$ is a orthogonal matrix and equation (3) determines a homothetic motion at $E^4_1$. (see [7])

(ii) For the case $\alpha = 1, \beta = -1$, $A$ is a semi-orthogonal matrix and equation (3) determines a homothetic motion in semi-Euclidean space $E^2_2$. (see [5]).
\end{special_cases}
Theorem 2. The derivation operator $\dot{B}$ of the Hamilton operator $B = hA$, is a quasi-orthogonal matrix.

Proof. We derivate of (2), i.e. $\dot{B} = H[\dot{a}(t)]$, we have $B^T \varepsilon B = \varepsilon$, and since $a(t)$ is unit velocity curve then $\det B = 1$. \hfill \Box

Theorem 3. In $E^4_{\alpha\beta}$, the Hamilton motion is a regular motion, and it does not depend on $h$.

If we differentiate of (3) with respect to $t$ yields

$$\dot{Y} = \dot{B}X + \dot{C} + B\ddot{X},$$

where

$$V_r = B\ddot{X}$$

is the relative velocity of $X$, $V_s = \dot{B}X + \dot{C}$ is the sliding velocity of $X$ and $V_a = \dot{Y}$ is called absolute velocity of point $X$. So, we can give the following theorem.

Theorem 4. In four-dimensional space $E^4_{\alpha\beta}$, for 1-parameter homothetic motion, absolute velocity vector of moving system of a point $X$ at time $t$ is the sum of the sliding velocity vector and relative velocity vector of that point.

5 Pole points and pole curves of the motion

We look for points where the sliding velocity of the motion is zero at all time $t$, such points are called pole points of the motion at that instant in $R_\circ$. Hence,

$$\dot{B}X + \dot{C} = 0. \quad (5)$$

by theorem 4.2, $\dot{B}$ is regular, so equation (5) has only one solution, i.e.

$$X = B^{-1} \cdot \dot{C}$$

at every instant $t$. In this case the following theorem can be given.

Theorem 5. The pole point corresponding to each instant $t$ in $R_\circ$ is the rotation by $B^{-1}$ of the speed vector $\dot{C}$ of the translation vector at that moment.

Proof. As the matrix $\dot{B}$ is quasi-orthogonal, the matrix $B^{-1}$ is quasi-orthogonal too. Thus, it makes a rotation. \hfill \Box

Theorem 6. During the homothetic motion the pole curves slide and roll upon each others and the number of the sliding-rolling of the motion is $h$. 
6 Acceleration centers of order \((r - 1)\) of the motion

**Definition 5.** The set of zeros of the equation of the sliding acceleration of order \(r\) is called the acceleration center of order \((r-1)\)[7].

In order to find the acceleration center of order \((r-1)\) for the equation (3) according to definition above, we find the solution of the equation

\[ B^{(r)}X + C^{(r)} = 0, \quad (6) \]

where

\[ B^{(r)} = \frac{d^r B}{dt^r}, \quad C^{(r)} = \frac{d^r C}{dt^r}. \]

As the curve \(a(t)\) is a regular curve of order \(r\), then

\[ (a^{(r)}_0(t))^2 + \alpha (a^{(r)}_1(t))^2 + \beta (a^{(r)}_2(t))^2 + \alpha\beta (a^{(r)}_3(t))^2 \neq 0, \quad a^{(r)}_i = \frac{d^r a_i}{dt^r}. \]

Also, as

\[ \det B^{(r)} = \left\{ [a^{(r)}_i]^2 + \alpha [a^{(r)}_1]^2 + \beta [a^{(r)}_2]^2 + \alpha\beta [a^{(r)}_3]^2 \right\}^2, \]

then \(\det B^{(r)} \neq 0\). Therefore matrix \(B^{(r)}\) has an inverse, and, by equation (6), the acceleration center of order \((r - 1)\) at every \(t\) instant, is

\[ X = [B^{(r)}]^{-1}(-C^{(r)}). \]

**Example 1.** Let \(a : I \subset \mathbb{R} \to E^4_{\alpha\beta}\) be a curve given by

\[ t \to a(t) = \frac{1}{\sqrt{2}} \left( \cos t, \frac{1}{\sqrt{\alpha}} \sin t, \frac{1}{\sqrt{\beta}} \cos t, \frac{1}{\sqrt{\alpha\beta}} \sin t \right), \quad \alpha, \beta \geq 0. \]

\(a(t)\) is a unit velocity curve and differentiable regular of order \(r\). Matrix \(B\) can be represented as

\[ B = H[a(t)] = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos t & -\sqrt{\beta} \sin t & -\sqrt{\alpha} \cos t & -\sqrt{\alpha\beta} \sin t \\ \frac{1}{\sqrt{\alpha}} \sin t & \cos t & -\sqrt{\beta} \sin t & \sqrt{\beta} \cos t \\ \frac{1}{\sqrt{\beta}} \cos t & \sqrt{\alpha} \sin t & \cos t & -\sqrt{\alpha} \sin t \\ \frac{1}{\sqrt{\alpha\beta}} \sin t & -\frac{1}{\sqrt{\beta}} \cos t & \frac{1}{\sqrt{\alpha}} \sin t & \cos t \end{bmatrix} \]

Thus \(a(t)\) satisfies all conditions of the above theorems.
let $C = (0, t, 0, 0)$, the (3) motion is given by

$$Y = \frac{1}{\sqrt{2}} \begin{bmatrix}
\cos t & -\sqrt{\alpha} \sin t & -\sqrt{\beta} \cos t & -\sqrt{\alpha \beta} \sin t \\
\frac{1}{\sqrt{\alpha}} \sin t & \cos t & -\sqrt{\frac{\alpha}{\beta}} \sin t & \sqrt{\beta} \cos t \\
\frac{1}{\sqrt{\beta}} \cos t & \sqrt{\frac{\beta}{\alpha}} \sin t & \cos t & -\sqrt{\alpha} \sin t \\
\frac{1}{\sqrt{\alpha}} \sin t & \frac{1}{\sqrt{\beta}} \cos t & \frac{1}{\sqrt{\alpha}} \sin t & \cos t
\end{bmatrix} \begin{bmatrix}
X \\
0 \\
t \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
t \\
0 \\
0
\end{bmatrix}. \quad (7)$$

Hence geometrical path of pole points in the Hamilton motion is determined by equation (7) as

$$X = B^{-1} \cdot (-C) = \varepsilon^{-1} B^T \cdot \varepsilon (-C)$$

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{\beta} \sin t \\
-\sqrt{\alpha} \frac{\beta}{\alpha} \cos t \\
\sin t \\
\frac{1}{\sqrt{\alpha}} \cos t
\end{bmatrix}.$$ 

References


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