Some Growth Properties Related to the Weak Type Entire Functions

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Abstract
In the paper we deduce some results on the growth properties based on the weak type of entire functions.

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1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. We use the standard notations and definitions in the theory of entire functions which are available in [5]. In the sequel we use the following notation:

$\log^k x = \log \left( \log^{k-1} x \right)$ for $k = 1, 2, 3, \ldots$ and $\log^0 x = x$.

The definition of order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are also well known. Datta and Jha[2] introduced the concept of weak type of an entire function. Its definition is as follows:

Definition 1 [2] The weak type $\tau_f$ of an entire function $f$ is defined as

$$
\tau_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.
$$
In the paper, using the concept of weak type we establish some results related to the growth properties of composite entire functions.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] If $f$ and $g$ are entire functions, for all sufficiently large values of $r$,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M\left(M\left(\frac{r}{2}, g\right), f\right).$$

Lemma 2 [3] Let $f$ be an entire function such that $0 < \lambda_f < \infty$. If $\tau_f$ and $\tau_{f^{(k)}}$ be the respective weak types of $f$ and $f^{(k)}$ then $\tau_{f^{(k)}} \leq (2^k)^{\lambda_f} \tau_f$ where $k = 0, 1, 2, 3, ...$

Lemma 3 [4] Let $f$ be an entire function of finite lower order. If there exist entire functions $a_i (i = 1, 2, 3, \ldots, n; n \leq \infty)$ satisfying

$$T\left(r, a_i\right) = o\{T\left(r, f\right)\} \text{ and } \sum_{i=1}^{n} \delta\left(a_i, f\right) = 1,$$

then

$$\lim_{r \to \infty} \frac{T(r, f)}{\log M(r, f)} = \frac{1}{\pi}.$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 Let $f$ and $g$ be two entire functions such that $0 < \lambda_f < \infty$ and $0 < \lambda_g < \infty$. Also let $0 < \tau_g < \infty$. Then

$$\limsup_{r \to \infty} \frac{\log^{[2]} M\left(r, f \circ g\right)}{\log M\left(r, g^{(k)}\right)} \geq \frac{\lambda_f}{2^{(k+1)} \lambda_g} \text{ for } k = 1, 2, 3, ...$$

Proof. Let $0 < \epsilon < \min\{\lambda_f, \tau_g\}$. Then for all sufficiently large values of $r$ we obtain that

$$\log M\left(\frac{r}{2}, g\right) > (\tau_g - \epsilon)(\frac{r}{2})^{\lambda_g}. \quad (1)$$

Again from the first part of Lemma 1, we get for all sufficiently large values of $r$ that

$$\log^{[2]} M\left(r, f \circ g\right) > (\lambda_f - \epsilon) \log \frac{1}{16} + (\lambda_f - \epsilon) \log M\left(\frac{r}{2}, g\right). \quad (2)$$
Now for all sufficiently large values of \( r \) it follows from (1) and (2) that
\[
\log[^2] M(r, f \circ g) > (\lambda_f - \epsilon) \log \frac{1}{16} + (\lambda_f - \epsilon)(\tau_g - \epsilon)(\frac{r}{2})^{\lambda_g}. \tag{3}
\]
Again by Lemma 2, we get for a sequence of values of \( r \) tending to infinity that
\[
\log M(r, g^{(k)}) < (\tau_g(k) + \epsilon) r^{\lambda_g(k)} \leq ((2^k)^{\lambda_g} \tau_g + \epsilon) r^{\lambda_g}. \tag{4}
\]
So from (3) and (4) it follows for a sequence of values of \( r \) tending to infinity that
\[
\frac{\log[^2] M(r, f \circ g)}{\log[^2] M(r, g^{(k)})} > \frac{(\lambda_f - \epsilon) \log \frac{1}{16} + (\lambda_f - \epsilon)(\tau_g - \epsilon)(\frac{r}{2})^{\lambda_g}}{(2^k \lambda_g \tau_g + \epsilon) r^{\lambda_g}}. \tag{5}
\]
Since \( \epsilon > 0 \) is arbitrary, we get from (5) that
\[
\limsup_{r \to \infty} \frac{\log[^2] M(r, f \circ g)}{\log[^2] M(r, g^{(k)})} \geq \frac{\lambda_f}{2^{(k+1)} \lambda_g}.
\]
This proves the theorem. \( \blacksquare \)

**Theorem 2** Let \( f \) and \( g \) be two entire functions with (i) \( \rho_f = \rho_g \) and (ii) \( 0 < \lambda_g \leq \rho_g < \infty \). Also, let there exist entire functions \( a_i \) \( (i = 1, 2, 3, \ldots, n; n \leq \infty) \) satisfying \( T(r, a_i) = o (T(r, g)) \) and \( \sum_{i=1}^{n} \delta(a_i, g) = 1 \). Then
\[
\min \{ \limsup_{r \to \infty} \frac{\log[^2] M(r, f \circ g)}{\log[^2] M(\exp(r^{\lambda_g}), f^{(k)})}, \limsup_{r \to \infty} \frac{\log[^2] M(r, f \circ g)}{\log[^2] M(\exp(r^{\lambda_g}), g^{(k)})} \} \geq (\frac{1}{2})^{\lambda_g} \pi \tau_g
\]
for \( k = 0, 1, 2, 3, \ldots \).

**Proof.** In view of Lemma 3 and the first part of Lemma 1, we obtain that
\[
\limsup_{r \to \infty} \frac{\log[^2] M(r, f \circ g)}{\log[^2] M(\exp(r^{\lambda_g}), f^{(k)})} \geq \limsup_{r \to \infty} \frac{\log[^2] M(\frac{1}{16} M(\frac{r}{2}, g), f)}{\log[^2] M(\exp(r^{\lambda_g}), f^{(k)})} = \limsup_{r \to \infty} \frac{\log M(\frac{r}{2}, g)}{\log \{\frac{1}{16} M(\frac{r}{2}, g)\}} \\
\lim_{r \to \infty} \frac{\log M(\frac{r}{2}, g)}{T(\frac{r}{2}, g)} \liminf_{r \to \infty} \frac{T(\frac{r}{2}, g)}{(\frac{r}{2})^{\lambda_g}}, \limsup_{r \to \infty} \frac{\log \{\exp(r^{\lambda_g})\}}{(\frac{1}{2})^{\lambda_g}} \geq \frac{\rho_f \pi \tau_g (\frac{1}{2})^{\lambda_g}}{\rho_f} = \pi \tau_g (\frac{1}{2})^{\lambda_g}. \tag{6}
\]
In a similar way exactly proceeding as above and in view of condition (i), we get that

\[
\limsup_{r \to \infty} \frac{\log^2 M(r, f\circ g)}{\log^2 M(\exp(r^\lambda_g), g^{(k)})} \geq \frac{\rho_f \pi \tau_g (1/2) \lambda_g}{\rho g} = \pi \tau_g (1/2) \lambda_g.
\]  

(7)

Thus the theorem follows from (6) and (7). ■

References


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