Min-Dom-Color Number of a Graph

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Abstract

Min-dom-color number \( md_\chi(G) \) is the minimum number of color classes which are dominating sets of \( G \), where the minimum is taken over all \( \chi \)-colorings of \( G \). In this paper we investigate graphs with \( md_\chi(G) = 0 \). We also prove certain necessary and sufficient conditions so that \( md_\chi(G) \) equals the chromatic number \( \chi(G) \). We conclude with the study of \( md_\chi(G) \) for cartesian product and semi-strong product of two graphs.

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1 Introduction

By a graph \( G = (V, E) \), we mean a connected, finite, undirected graph with neither loops nor multiple edges. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. For graph theoretic terminology we refer to Harary [2]. Graph coloring theory and domination in graphs are two major areas within graph theory which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number \( \chi(G) \) of a graph \( G \) which is defined to be the minimum number of colors required to color the vertices of \( G \) in such a way that no two adjacent vertices receive the same color. If \( \chi(G) = k \), we say that \( G \) is \( k \)-chromatic. The chromatic number is a very well studied parameter whose history dates back to the famous four color problem and the early work of Kempe [6] and Heawood [4] in 1890.

Another fastest growing area within graph theory is the study of domination and related subset problems such as independence, covering and matching. A comprehensive treatment of the fundamentals of domination is given in Haynes
et al. [3]. Let \( G \) be a \( k \)-chromatic graph. A set \( S \subseteq V \) is said to be a dominating set of \( G \) if every vertex \( v \) in \( V - S \) is adjacent to a vertex in \( S \). Color dominating vertex is a vertex having a neighbor in each of the other \( k-1 \) classes.

In this paper we define min-dom-color number of a graph \( G \) and study graphs \( G \) with \( \text{md}_\chi(G) = 0 \). Further, we prove certain sufficient conditions and characterisations for graphs with \( \text{md}_\chi(G) = \chi(G) \). We conclude with results on \( \text{md}_\chi(G) \) when \( G = H \times K \) where \( \times \) is cartesian product, semi-strong product or join of \( H \) and \( K \).

We need the following definitions.

**Definition 1.1.** [1] Let \( G \) be a graph with \( \chi(G) = k \). Let \( C = V_1, V_2, \ldots, V_k \) be a \( k \)-coloring of \( G \). Let \( d_C \) denote the number of color classes in \( C \) which are dominating sets of \( G \). Then \( \text{md}_\chi(G) = \min_C d_C \), where the minimum is taken over all \( k \)-colorings of \( G \), is called the **dom-color number** of \( G \).

**Definition 1.2.** [2] Let \( G_1 \) and \( G_2 \) be two graphs. Cartesian product \( G_1 \times G_2 \) is the graph having vertex set \( V(G_1 \times G_2) = V(G_1) \times V(G_2) \) and edge set \( E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2)/u_1u_2 \in E(G_1) \text{ and } v_1 = v_2 \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)\} \).

The sum \( G_1 + G_2 \) is the graph having vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \) together with all the edges joining the points of \( V(G_1) \) to the points of \( V(G_2) \).

**Definition 1.3.** [5] Semi-strong product \( G_1 \circ G_2 \) is the graph having vertex set \( V(G_1 \circ G_2) = V(G_1) \times V(G_2) \) and edge set \( E(G_1 \circ G_2) = \{(u_1, v_1)(u_2, v_2)/u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2) \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)\} \).

### 2 Main Results

**Definition 2.1.** Let \( G \) be a graph with \( \chi(G) = k \). Let \( C = V_1, V_2, \ldots, V_k \) be a \( k \)-coloring of \( G \). Let \( d_C \) denote the number of color classes in \( C \) which are dominating sets of \( G \). Then \( \text{md}_\chi(G) = \min_C d_C \), where the minimum is taken over all \( k \)-colorings of \( G \), is called the **min-dom-color number** of \( G \).

Min-dom-color number of \( G \) exists for all graphs and \( 0 \leq \text{md}_\chi(G) \leq \chi(G) \). The following are immediate.

**Proposition 2.2.** 1. \( \text{md}_\chi(K_n) = n \).
2. \( md\chi(C_n) = \begin{cases} 
0 & \text{if } n \geq 9 \text{ and } n \text{ is odd} \\
1 & \text{if } n = 7 \\
2 & \text{if } n \text{ is even or } n = 5 \\
3 & \text{if } n = 3 
\end{cases} \)

For \( C_n \), \( n \geq 9 \) and \( n \) odd, \( P_8 \) is an induced subgraph which can be colored by the sequence 1, 2, 1, 2, 1, 2, 3, 2, 3, 1, 3. This coloring of \( P_8 \), when extended to \( C_n \) gives a chromatic partition having no dominating sets and hence the result follows.

3. \( md\chi(W_n) = md\chi(C_n) + 1 \).
4. For any complete \( k \)-partite graph \( G \), \( md\chi(G) = k \).
5. For any uniquely colorable graph \( G \), \( md\chi(G) = \chi(G) \). In particular, for a tree \( T \), \( md\chi(T) = 2 \).
6. If \( G \) is a bipartite graph without isolated vertices then \( md\chi(G) = 2 \).
7. If \( G \) is a graph with at least two isolated vertices, then \( md\chi(G) = 0 \).
8. If \( \Delta(G) = n - 1 \), then \( md\chi(G) \geq 1 \).

**Proposition 2.3.** Let \( G \) be any graph. If \( V(G) \) can be partitioned into two sets \( U \) and \( W \) such that \( md\chi(<U>) = \chi(G) \), \( \chi(<W>) = \chi(G) \) and \( md\chi(<W>) \geq k \) then \( md\chi(G) \geq k \).

**Proof.** Any \( \chi \)-chromatic partition of \( G \) is of the form \( \{U_1 \cup W_1, U_2 \cup W_2, \ldots, U_\chi \cup W_\chi\} \) where \( U_i \subset U \) and \( W_i \subset W \). Then \( \{W_1, W_2, \ldots, W_\chi\} \) is a chromatic partition of \( <W> \) and \( \{U_1, U_2, \ldots, U_\chi\} \) is a chromatic partition of \( <U> \). Since \( md\chi(<U>) = \chi \) and \( md\chi(<W>) \geq k \), every \( U_i \) (\( 1 \leq i \leq \chi \)) is a dominating set of \( <U> \) and at least \( k \) \( W_i \)'s say \( W_{i_1}, W_{i_2}, \ldots, W_{i_k} \) are dominating sets of \( <W> \). Then \( U_{i_1} \cup W_{i_1}, U_{i_2} \cup W_{i_2}, \ldots, U_{i_k} \cup W_{i_k} \) are dominating sets of \( G \). Hence \( md\chi(G) \geq k \).

We now prove certain sufficient conditions so as to have \( md\chi(G) = 0 \).

**Theorem 2.4.** Let \( G \) be a graph with \( \chi(G) \geq 4 \). If \( G \) has two supports \( u \) and \( v \) which are either adjacent or nonadjacent with \( \chi(G + uv) = \chi(G) \), then \( md\chi(G) = 0 \).

**Proof.** Choose a chromatic partition \( C \) of \( G \) where the supports are in different sets. Choose two color classes in \( C \) which do not contain either \( u \) or \( v \) and add a pendent vertex adjacent to \( u \) and a pendent vertex adjacent to \( v \) in those classes. The obtained chromatic partition then has no dominating sets. \( \square \)
Theorem 2.5. If $\chi(G) \geq 3$ and $G$ has an induced subgraph $P_8$ with $\deg_G v = 2$ for all internal vertices of $P_8$, then $md_\chi(G) = 0$.

Proof. Let $v_1, v_2, \ldots, v_8, v_9$ be the set of vertices whose induced subgraph satisfies our hypothesis. Choose a chromatic partition of $G$. If $v_1$ and $v_9$ have the same color say 1 then recolor the vertices $v_2, v_3, v_4, v_5, v_6, v_7$ and $v_8$ with colors 2, 3, 2, 1, 2, 1 and 3 respectively. If $v_1$ and $v_2$ have different colors, say 1 and 2 then recolor the vertices $v_2, v_3, v_4, v_5, v_6, v_7$ and $v_8$ with colors 3, 1, 3, 2, 3, 2, 1 respectively. Then no color class of the resulting chromatic coloring dominates all the vertices $v_1, v_2, \ldots, v_7$ and $v_8$. Hence $md_\chi(G) = 0$. \hfill \Box

Theorem 2.6. If $\chi(G) \geq 4$ and $G$ has an induced subgraph $P_6$ with $\deg_G v = 2$ for all internal vertices of $P_6$ then $md_\chi(G) = 0$.

Proof. Let $v_1, v_2, v_3, \ldots, v_6, v_7$ be the vertices in $G$ whose induced subgraph satisfies our hypothesis. Choose any chromatic coloring of $G$. If $v_1$ and $v_7$ have the same color say 1 or they have different colors 1 and 2, recolor $v_2, v_3, v_4, \ldots, v_5$ and $v_6$ with colors 3, 2, 4, 1, 3 respectively. Then no color class in the obtained chromatic partition dominates all the vertices $v_1, v_2, \ldots, v_6$. Hence $md_\chi(G) = 0$. \hfill \Box

Theorem 2.7. If $\chi(G) \geq 5$ and $G$ has an induced subgraph $P_5$ with $\deg_G v = 2$ for all internal vertices of $P_5$ then $md_\chi(G) = 0$.

Proof. Let $v_1, v_2, \ldots, v_6$ be the vertices in $G$ whose induced subgraph satisfies our hypothesis. Fix a chromatic partition of $G$. $v_1$ and $v_6$ may get the same color say 1 or different colors say 1 and 2. Recoloring $v_2, v_3, v_4$ and $v_5$ by colors 2, 3, 4 and 5 we obtain a chromatic partition where none of the sets are dominating. \hfill \Box

Theorem 2.8. Let $G$ be a graph with $md_\chi(G) = d_\chi(G)$. Then any color class in any chromatic partition of $G$ has at most one non color dominating vertex.

Proof. Let $A$ be a color class in any chromatic partition of $G$. case 1. $A$ is a dominating set.

Suppose that there is a non color dominating vertex $v$ in $A$. Then there is a color class $B$ such that $v$ is not adjacent to any of the vertices of $B$. As $B$ is not a dominating set and $md_\chi(G) = d_\chi(G)$ moving $v$ from $A$ to $B$ should make $B$ a dominating set. Thus there can be no other non color dominating vertex in $A$ which is not adjacent to any of the vertices in $B$. If $A$ has one more non color dominating vertex $u \neq v$ then there exists another color class $C$ such that no vertex of $C$ is adjacent to $u$. Now moving $v$ to $B$ and $u$ to $C$ makes $B$ and $C$ dominating sets, thereby increasing the number of dominating sets in the newly obtained chromatic partition, which is not possible as $md_\chi(G) = d_\chi(G)$. 

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Hence $A$ has at most one non color dominating vertex.

case 2. $A$ is not a dominating vertex. Suppose that $v \in A$ is a non color dominating vertex. Then there is a non dominating color class $B$ such that no vertex of $B$ is adjacent to $v$. Since $A$ is not a dominating set and $md_\chi(G) = d_\chi(G)$ it must have a vertex $u$ in some dominating color class so that no vertex of $A$ is adjacent to $u$; if not then moving all vertices which are not dominated by $A$ to $A$ makes $A$ a dominating set, thereby increasing the number of dominating sets. Now moving $v$ to $B$ and $u$ to $A$ we obtain a chromatic partition in which the total number of dominating sets is decreased by one. This contradiction shows that $A$ has no non color dominating vertex.

**Theorem 2.9.** Let $G$ be a graph. Then $md_\chi(G) = d_\chi(G) = 1$ if and only if $md_\chi(G - v) = \chi(G)$ where $v$ is an isolated vertex of $G$.

**Proof.** Let $\{V_1, V_2, \ldots, V_\chi\}$ be a chromatic partition of $G$. Without loss of generality, assume that $V_1$ is the dominating set. For $2 \leq j \leq \chi(G)$, if $V_j$ dominates all the vertices of $V_1$ then let $V'_j$ be obtained by adding the vertices not dominated by $V_j$ into $V_j$. Now we obtain a chromatic partition with two dominating sets $V_1$ and $V'_j$ which is not possible since $d_\chi(G) = 1$. Hence $V_j$ does not dominate all the vertices of $V_1$. Then there exists $v_j \in V_1$ such that no vertex of $V_j$ is adjacent to $v_j$. Now $\{V_1 - \{v_j\}, V_2, \ldots, V_{j-1}, V_j \cup \{v_j\}, V_{j+1}, \ldots, V_\chi\}$ is a chromatic partition of $G$. Since $V_1 - \{v_j\}$ is not a dominating set and $md_\chi(G) = 1$, $V_j \cup \{v_j\}$ must be a dominating set in $G$. Hence $V_j$ must dominate all other sets except $V_1$ for all $2 \leq j \leq \chi$. Suppose that $v_l \neq v_l$, $j, l \in \{2, 3, \ldots, \chi\}$. Then $\{V_1 - \{v_j, v_l\}, V_2, \ldots, V_{j-1}, V_j \cup \{v_j\}, V_{j+1}, \ldots, V_{l-1}, V_l \cup \{v_l\}, V_{l+1}, \ldots, V_\chi\}$ is a chromatic partition where $V_j \cup \{v_j\}$ and $V_l \cup \{v_l\}$ are dominating sets which is not possible. Hence $v_2 = v_3 = \cdots = v_\chi = v$ (say). This implies that $v$ is not adjacent to any of the vertices in $V_2, V_3, \ldots, V_\chi$ so that $v$ is an isolated vertex. We claim that $md_\chi(G - v) = \chi(G)$. If not, then there is a chromatic partition of $G - v$, $\{U_1, U_2, \ldots, U_\chi\}$ such that $U_1$ is not a dominating set of $G - v$. Then $\{U_1 \cup \{v\}, U_2, U_3, \ldots, U_\chi\}$ is a chromatic partition of $G$ with no dominating sets, which is a contradiction. This contradiction proves that $md_\chi(G - v) = \chi(G)$.

The following are some sufficient conditions so that $md_\chi(G)$ equals its upper bound $\chi(G)$.

**Theorem 2.10.** Let $G$ be a graph with $\delta(G) \geq \chi(G) - 1$ and the number of edges between any two color classes in any chromatic partition varies at most by one. Then $md_\chi(G) = \chi(G)$.

**Proof.** Choose any chromatic partition $\{V_1, V_2, \ldots, V_\chi\}$ of $G$. If $V_i$ is not a dominating set we can choose a vertex $v$ in $V_j$ such that no vertex of $V_i$
is adjacent to $V$. As $\delta(G) \geq \chi(G) - 1$ we can choose a set $V_k$ such that $v$ is adjacent to at least two vertices in $V_k$. Now $\{V_1, \ldots, V_{i-1}, V_i \cup \{v\}, V_{i+1}, \ldots, V_j - \{v\}, V_{j+1}, \ldots, V_b\}$ is also a chromatic partition such that the number of edges between $V_i \cup \{v\}$ and $V_k$ varies by more than 2 edges. This contradiction shows that in any chromatic partition all the sets are dominating sets.

Let $s(G)$ denote the minimum cardinality of any set in any $\chi$-chromatic partition of $G$ and let $t(G)$ denote the maximum cardinality of any set in any $\chi$-chromatic partition of $G$, where the minimum and maximum are taken over all possible $\chi$-chromatic partitions of $G$.

**Proposition 2.11.** If $G$ is any graph with $s(G) = t(G)$, then $md_\chi(G) = \chi(G)$.

**Proof.** Let $C$ be any $\chi$-chromatic partition of $G$ and let $D \in C$. If $D$ is not a dominating set of $G$, then there exists a vertex $v$ which is not dominated by $D$. It is now possible to get another $\chi$-chromatic partition with the set $D \cup \{v\}$ having cardinality $|D| + 1$. But this is impossible since $s(G) = t(G)$ means each color class in any $\chi$-chromatic partition has the same cardinality.

We now characterise graphs with $md_\chi(G) = \chi(G)$.

**Theorem 2.12.** For any graph $G$, $md_\chi(G) = \chi(G)$ if and only if every vertex of $G$ lies in some induced subgraph $U$ of $G$ with $md_\chi(<U>) = \chi(G)$.

**Proof.** Necessity is obvious. To prove sufficiency, assume that every vertex of $G$ lies in some induced subgraph $U$ of $G$ with $md_\chi(<U>) = \chi(G)$. Let $\{V_1, V_2, \ldots, V_b\}$ be any $\chi$-chromatic partition of $G$. It is enough to prove that $V_i$ is a dominating set for all $i$. If $v \in V_j$ for some $j \neq i$, then by assumption $v$ lies in some induced subgraph with $md_\chi(<U>) = \chi(G)$. Since $\{V_1 \cap U, V_2 \cap U, \ldots, V_b \cap U\}$ is a chromatic partition of $<U>$, $V_i \cap U$ is a dominating set in $<U>$ so that $v$ is adjacent to some vertex of $V_i \cap U$. Hence $v$ is adjacent to some vertex of $V_i$ and so each $V_i$ is a dominating set of $G$. Therefore $md_\chi(G) = \chi(G)$.

**Corollary 2.13.** If every vertex $v$ of $G$ lies in some uniquely $\chi$-colorable induced subgraph of $G$, then $md_\chi(G) = \chi(G)$.

Converse of the above corollary is not true. For the graph $G$ given in Figure 1, $md_\chi(G) = \chi(G)$ but yet the vertex $v$ lies in no uniquely 3-colorable induced subgraph of $G$. 
Theorem 2.14. Let $G$ be a $\chi$-critical graph. Then $md_\chi(G) = \chi(G)$ if and only if $G$ is complete.

Proof. Suppose that $md_\chi(G) = \chi(G)$. Let $v \in V(G)$. Since $G$ is $\chi$-critical, there is a chromatic partition $\{\{v\}, V_2, V_3, ..., V_k\}$. Since $md_\chi(G) = \chi(G)$, $\{v\}$ is a dominating set. This implies that $deg_G v = n-1$. Thus $deg_G v = n-1 \forall v \in V(G)$. Hence $G$ is complete. Converse is evident.

Theorem 2.15. Let $G_1 \times G_2$ be the cartesian product of two graphs $G_1$ and $G_2$.

i) If $\chi(G_2) < \chi(G_1)$, then $md_\chi(G_1 \times G_2) = \chi(G_1 \times G_2)$ if and only if $md_\chi(G_1) = \chi(G_1)$.

ii) If $\chi(G_1) = \chi(G_2)$, then $md_\chi(G_1 \times G_2) = \chi(G_1 \times G_2)$ if and only if $md_\chi(G_1) = \chi(G_1)$ or $md_\chi(G_2) = \chi(G_2)$.

Proof. Let $\max \{\chi(G_1), \chi(G_2)\} = \chi(G_1)$. Let $\{V_1, V_2, ..., V_{\chi(G_1)}\}$ be a $\chi(G_1)$-chromatic partition of $G_1$ and $\{U_1, U_2, ..., U_{\chi(G_1)}\}$ be a proper $\chi(G_1)$-coloring of $G_2$. Let $W_k = \{(x, y)/x \in V - i, y \in U_j, i - j \equiv (k-1)mod \chi(G_1)\}$ where $k = 1, 2, ..., \chi(G_1)$. Let $x_1 \times y_1, x_2 \times y_2 \in W_k, k \in \{1, 2, ..., \chi(G_1)\}$ where $x_1 \in V_{i_1}, x_2 \in V_{i_2}, y_1 \in U_{j_1}$ and $y_2 \in U_{j_2}$. We have $i_1 - j_1 \equiv (i_2 - j_2)(mod \chi(G_1)) \text{ or } i_1 - j_1 \equiv (j_1 - j_2)(mod \chi(G_1))$ which implies $i_1 \neq i_2 \leftrightarrow j_1 \neq j_2 - - - - - - - (1)$

$x_1 \times y_1$ and $x_2 \times y_2$ are adjacent if and only if $x_1 = x_2$ and $y_1$ is adjacent to $y_2$ or $x_1$ is adjacent to $x_2$ and $y_1 = y_2$. This happens only if $i_1 \neq i_2$ and $j_1 = j_2$ or $j_1 \neq j_2$ and $i_1 = i_2$, which is impossible by (1). Hence $x_1 \times y_1$ and $x_2 \times y_2$ are non-adjacent. Thus each of $W_i$'s are independent sets. Also each element $x \times y \in G_1 \times G_2$ where $x \in V_i$ and $y \in U_j$ belongs to exactly one of the set namely $W_{[i-j]mod \chi(G_1)+1}$ so that $W_k$ form a partition of the vertex set of $G_1 \times G_2$. Then $\{W_1, W_2, ..., W_{\chi(G_1)}\}$ is a proper $\chi(G_1)$-coloring of $G_1 \times G_2$.

Assume, without loss of generality, $md_\chi(G_1) = \chi(G_1) = \max \{\chi(G_1), \chi(G_2)\}$. Let $x \times y \in G_1 \times G_2$. Then $x \times y$ lies in an induced subgraph $H$ of $G_1 \times G_2$ which is isomorphic to $G_1$. Now any $\chi(G_1)$-chromatic partition of $G_1 \times G_2$ induces a $\chi(G_1)$-chromatic partition on $H$ as $md_\chi(G_1) = \chi(G_1)$. $x \times y$ is a color dominating vertex in $H$ and so in $G_1 \times G_2$. Thus every vertex of $G_1 \times G_2$ is a color dominating vertex. Hence $md_\chi(G_1 \times G_2) = \chi(G_1 \times G_2)$.
Conversely assume \( md_\chi(G_1 \times G_2) = \chi(G_1 \times G_2) \). Suppose \( md_\chi(G_1) < \max \{ \chi(G_1), \chi(G_2) \} \) and \( md_\chi(G_2) < \max \{ \chi(G_1), \chi(G_2) \} \). Let \( \max \{ \chi(G_1), \chi(G_2) \} = t \). Since \( md_\chi(G_1) < t \), there exists a proper coloring \( \{ V_1, V_2, \ldots, V_t \} \) of \( G_1 \) such that there is a vertex \( a \in V_2 \) which is not adjacent to any vertex of \( V_1 \). Similarly, since \( md_\chi(G_2) < t \) there exist a proper coloring of \( G_2 \) say \( \{ U_1, U_2, \ldots, U_t \} \) such that there is \( b \in U_1 \) which is not adjacent to any vertex of \( U_2 \). Let \( W_k = \{ x \times y \in V_i, y \in V_j, t - j \equiv k - 1(mod t) \} \) \( k = 1, 2, 3, \ldots, t \). As above \( W_1, W_2, \ldots, W_t \) is a \( t \)-chromatic partition of \( G_1 \times G_2 \). \( a \times b \) is adjacent only to the vertices of the form \( a \times y \) where \( y \) is adjacent to \( b \) and \( x \times b \) where \( x \) is adjacent to \( a \). We know that \( W_1 = (V_1 \times U_1) \cup (V_2 \times U_2) \cup \ldots \cup (V_t \times U_t) \). Since no vertex of \( U_2 \) is adjacent to \( b \), no vertex of \( V_2 \times U_2 \) is adjacent to \( a \times b \). Also no vertex of \( V_1 \) is adjacent to \( a \) and so no vertex of \( V_1 \times U_1 \) is adjacent to \( a \times b \). Therefore no vertex of \( W_1 \) is adjacent to \( a \times b \). Hence \( W_1 \) is not a dominating set in \( G_1 \times G_2 \), which is a contradiction to the fact that \( md_\chi(G_1 \times G_2) = \chi(G_1 \times G_2) \). Hence our assumption is wrong and so at least one of \( md_\chi(G_1) = \max \{ \chi(G_1), \chi(G_2) \} \) or \( md_\chi(G_2) = \max \{ \chi(G_1), \chi(G_2) \} \) holds. \( \square \)

**Theorem 2.16.** \( md_\chi(G_1 + G_2) = md_\chi(G_1) + md_\chi(G_2), \) \( d_\chi(G_1 + G_2) = d_\chi(G_1) + d_\chi(G_2) \)

**Proof.** Proof follows immediately from the fact that \( A \) is a chromatic partition of \( G_1 + G_2 \) if and only if \( A = B \cup C \) where \( B \) and \( C \) are chromatic partitions of \( G_1 \) and \( G_2 \). \( \square \)

**Theorem 2.17.** Let \( G_1 \) and \( G_2 \) be two graphs. Then \( \min_\chi(G_1 \circ G_2) \leq md_\chi(G_2) \) and \( d_\chi(G_1 \circ G_2) \geq d_\chi(G_2) \).

**Proof.** Choose a chromatic partition \( \{ V_1, V_2, \ldots, V_{md_\chi(G_2)}, \ldots, V_{\chi(G_2)} \} \) of \( G_2 \) where all \( V_i \) with \( i \leq \min_\chi(G_2) \) are dominating sets and all \( V_i \) with \( i > \min_\chi(G_2) \) are not dominating sets. Let \( W_1 = \{(u, v)/u \in V_i, 1 \leq i \leq \chi \} \). Since \( V_i \) are independent, \( W_i \) are independent. Hence \( \{ W_1, W_2, \ldots, W_{\chi(G_2)} \} \) is a chromatic partition of \( G_1 \circ G_2 \). Since all \( V_i \) with \( i > \min_\chi(G_2) \) are not dominating sets, there exist vertices \( v_i \) which are adjacent to no vertices in \( V_i \) for all \( i > \min_\chi(G_2) \). And so no vertex in \( W_i \) is adjacent to any vertex of the form \( u \times v_i \). So all \( W_i \), \( i > \min_\chi(G_2) \) are not dominating sets, so that \( \min_\chi(G_1 \circ G_2) \leq \min_\chi(G_2) \). Proof of the second part is analogous. \( \square \)

**Corollary 2.18.** For any two graphs \( G_1 \) and \( G_2 \) \( md_\chi(G_1 \circ G_2) = \chi(G_1 \circ G_2) \) if and only if \( md_\chi(G_2) = \chi(G_2) \).

**Proof.** Necessity follows from Theorem 2.17. To prove sufficiency, assume \( md_\chi(G_2) = \chi(G_2) \). Any vertex of \( G_1 \circ G_2 \) lies in an induced subgraph of \( G_1 \circ G_2 \) isomorphic to \( G_2 \). As \( \chi(G_1 \circ G_2) = \chi(G_2) \), by Theorem 2.12, \( md_\chi(G_1 \circ G_2) = \chi(G_1 \circ G_2) \). \( \square \)
3. Open problems

1. Characterise the class of graphs with $md_{\chi}(G) = 0$.
2. Characterise the class of graphs with $md_{\chi}(G) = \chi(G)$.
3. Characterise the class of graphs with $md_{\chi}(G) = d_{\chi}(G)$.

References


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