A New Derivation of Taylor’s Results in the Motion of Solids in a Fluid which Rotates Uniformly

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Abstract

The two-dimensional motion of an incompressible, uniformly rotating fluid past a cylindrical body of any cross-section is considered when the generators of the cylindrical body are perpendicular to the planes of fluid motion and the cylindrical body itself moves in an assigned manner through the fluid. Taylor found the extra forces of reaction of fluid motion on the cylinder, and also the extra moment of these forces on the cylinder about the center of mass, due to the rotation of the fluid. Here we rederive the results of Taylor by a different, simple and shorter method based on theory of Mechanics. The present method enables us to conclude that the previous results of Taylor, obtained for an incompressible fluid of constant density, cannot be extended to the case of a compressible fluid.

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1 Introduction

Taylor \cite{5} considered the two-dimensional flow of an incompressible, inviscid fluid of uniform density in parallel planes perpendicular to the generators of an
infinite solid cylinder which moved through the fluid in an assigned manner, starting from rest, the fluid having a given irrotational motion at infinity. He calculated the additional force of reaction of fluid motion on the solid cylinder when a uniform angular velocity of rotation, with the axis of rotation parallel to the generators of the cylinder, is imposed on the whole system. Taylor also considered the additional moment of the force of reaction of the fluid motion, about the center of mass of the cylinder, due to this rotation. In his detailed analysis Taylor used the stream function of fluid motion, since he considered essentially a two-dimensional flow of an incompressible fluid of uniform density. In this paper we rederive Taylor’s results by a shorter and alternative method using the basic ideas of Mechanics. We also examine to what extent Taylor’s results can be extended to a compressible fluid.

2 Additional force of reaction of fluid motion and its moment due to rotation

We consider an inertial frame of reference \((\xi, \eta, \zeta)\) and choose a coordinate system \((x, y, z)\) rotating with angular velocity \(\omega\) about the \(z\)-axis. The \(z\)-axis is taken to coincide with the \(\zeta\)-axis (see Figure 1).

\[\text{Figure 1: Schematic diagram of the flow past a cylinder moving in a liquid with velocity } \vec{\omega} \text{ about the } z\text{-axis. } R \text{ is the position vector of centre of mass of the cylinder, } S \text{ is cross-section and } s \text{ is contour of the cross-section.}\]

The acceleration in the two frames are related to each other \([2]\) through the
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\[
\left( \frac{du^i}{dt} \right)^0 = \frac{d\vec{u}}{dt} + 2\vec{\omega} \times \vec{u} - \omega^2 \vec{r},
\]

(1)

where the superscript 0 denotes quantities taken in the inertial frame. Thus \( \left( \frac{du^i}{dt} \right)^0 \) is the acceleration of a fluid particle as observed in the inertial frame of reference and \( \vec{u}^0 \) is the velocity in that frame. Similarly, \( \frac{d\vec{u}}{dt} \) is the acceleration of the same fluid particle in the rotating frame of reference where \( \vec{u} \) is its velocity in that frame of reference. The terms \( 2\vec{\omega} \times \vec{u} \) and \( -\omega^2 \vec{r} \) in (1) give the Coriolis acceleration and centrifugal force, respectively, when the angular velocity of rotation \( \vec{\omega} = (0, 0, \omega) \) is constant. Euler’s equation in the rotating frame of reference, in the usual notation, takes the form (cf. (1))

\[
\frac{d\vec{u}}{dt} = -2\vec{\omega} \times \vec{u} + w^2 \vec{r} - \frac{1}{\rho} \nabla p
\]

(2)

Note that the fluid may be compressible in general.

In the further discussion we shall use the rotating frame of reference when a uniform angular velocity of rotation \( (0, 0, \omega) \) is imposed on the flow. In order to calculate the extra force acting on the moving cylindrical body when the additional rotation \( \vec{\omega} \) is imposed on the flow without rotation, we must ensure that in both the inertial and the rotating system the same flow may occur. This becomes clear by considering the dynamical equation. Taking the curl of (2), we have

\[
\frac{d}{dt}(\text{curl} \ \vec{u}) + (\text{curl} \ \vec{u}) \text{ div} \ \vec{u} = -2\vec{\omega} \text{ div} \ \vec{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p,
\]

(3)

We shall discuss the two cases of compressible and incompressible fluids separately.

If we consider the fluid to be compressible, in which \( p \) is a function of \( \rho \), equation (3) is simplified to

\[
\frac{d}{dt}(\text{curl} \ \vec{u}) + (\text{curl} \ \vec{u}) \text{ div} \ \vec{u} = -2\omega \text{ div} \ \vec{u}.
\]

(4)

If, in the absence of rotation \( (\vec{\omega} = 0) \), the same flow \( \vec{u} \) is to occur, then

\[
\frac{d}{dt}(\text{curl} \ \vec{u}) + (\text{curl} \ \vec{u}) \text{ div} \ \vec{u} = 0
\]

(5)

should be satisfied. Comparing (4) and (5), we conclude that

\[
\text{div} \ \vec{u} = 0
\]

(6)
should be satisfied, if \( \omega \neq 0 \). Thus a necessary condition that the dynamical equations in both inertial and rotating frames of references be identical, and hence the same flow \( \vec{u} \) occurs in both frames, is that the fluid is incompressible (cf. eq. (6)).

We next examine if this condition of incompressibility is also a sufficient condition for the flow to occur in inertial and rotating frames of reference. For an incompressible flow (\( \text{div} \vec{u} = 0 \)), eq. (3) is reduced to

\[
\frac{d}{dt}(\text{curl} \vec{u}) = \frac{1}{\rho^2} \nabla p \times \nabla \rho
\]  

(7)

As, in general, the pressure will be modified due to the imposition of rotation, the right hand side of the dynamical equation (7) will be different in the two cases of flows having rotation or not. The same flow \( \vec{u} \) cannot be maintained in the two cases. The case of an incompressible flow of uniform density is, however, different. In this case, considered earlier by Taylor, the right hand side of (7) vanishes, and the dynamical equation do not change from inertial frame of reference to the rotating frame. We shall confine ourselves in the further discussion to this case of an incompressible fluid of uniform density.

In the following we shall compare external forces acting on the moving body in rotating and nonrotating incompressible flows. \( \vec{F} \), the force of reaction due to fluid pressure per unit length of the solid cylinder is given at any time, by the equation

\[
\vec{F} = -\int_s p \vec{n} ds
\]  

(8)

where \( \vec{n} \) is the unit normal vector, directed away from the solid cylinder, and \( ds \) is an element of length of the bounding contour \( s \) of the normal cross-section \( S \) of the solid cylinder and the line integral in (8) is taken over the contour. We shall find an expression for \( \vec{F} \).

Let us assume that the rigid cylinder is momentarily removed and the fluid flow is extended to the region previously occupied by this cylinder in such a way that the contour \( s \) of the (removed) solid cylinder cross-section \( S \) is a stream line so that no fluid crosses it. It is clear that the flow outside the cross-section \( S \) remains unchanged in the extended flow and the force acting on the liquid cylinder inside \( S \) due to the flow outside is same as that acting on the solid cylinder due to the outside flow.

In our further discussion, we shall use the following theorem [1]:

If the flow is two-dimensional, \( \phi \) is a field function, \( \psi = \int \rho \phi dS \) then

\[
\frac{d\psi}{dt} = \int \rho \frac{d\phi}{dt} dS
\]  

(9)

Here \( \phi \) can be either a vector or a scalar field and the integral is over some region of the flow.
We consider the total linear momentum of the fluid which occupies the region $S$ vacated by the solid cylinder on its removal. The linear momentum in the inertial frame is given as

$$M^0 = \int \rho \mathbf{u}^0 \, dS = \int \rho \left( \frac{dr}{dt} \right)^0 \, dS,$$

(10)

where $\mathbf{u}^0$ is the fluid velocity at $\mathbf{r}^*$ using (9), we can write, in view of (10), the result

$$M^0 = \frac{d}{dt} \left( \int \rho \mathbf{r} \, dS \right)^0 = \frac{d}{dt} (M \mathbf{R})^0 = M \left( \frac{d\mathbf{R}}{dt} \right)^0 = M \mathbf{U}^0,$$

(11)

where $M$ is the total mass of fluid within $S$ and $\mathbf{R}$ and $\mathbf{U}^0$ are the position vector and velocity of the centre of mass of this fluid in inertial frame of reference. Since the contour $s$ of $S$ is a stream line across which no flow of fluid takes place, the mass $M$ is a constant. Also, the total external force $\mathbf{F}$ acting on the fluid within $S$, in the presence of rotation $\mathbf{\omega}$, which is also the force of reaction of the fluid flow outside $S$, is equal to the rate of change of momentum $M^0$. We can therefore write

$$\mathbf{F} = \left( \frac{dM^0}{dt} \right)^0.$$

(12)

But, in view of (11),

$$\left( \frac{dM^0}{dt} \right)^0 = M \left( \frac{d\mathbf{U}^0}{dt} \right)^0.$$

(13)

Using (1), we can write, in view of (12) and (13), the equation

$$\mathbf{F} = M \left( \frac{d\mathbf{U}}{dt} + 2 \mathbf{\omega} \times \mathbf{U} - \mathbf{\omega}^2 \mathbf{R} \right),$$

where $\mathbf{U}$ and $\frac{d\mathbf{U}}{dt}$ are the velocity and acceleration of the centre of mass of fluid within $S$ in the frame of coordinates which rotates with angular velocity $\omega$ about the $\zeta$-axis. In the absence of rotation ($\omega = 0$), the force experienced by the cylinder is therefore given by

$$\mathbf{F}' = M \frac{d\mathbf{U}}{dt}.$$

Hence, the additional force of reaction of fluid motion, acting on the solid arising from rotation is given by

$$\mathbf{F} - \mathbf{F}' = M (2 \mathbf{\omega} \times \mathbf{U} - \omega^2 \mathbf{R}).$$
This result, derived here for an incompressible fluid was also found by Taylor [5]. He used stream function of fluid motion in deriving his result. In our derivation, however, we have used a shorter and different method, based on the principles of mechanics.

We next consider the additional moment of the force of reaction due to fluid motion on the solid cylinder (per unit length) that arises from the imposition of a uniform rotation on the whole system of nonrotating flow. The moment of this external force of reaction, taken about the center of mass of fluid within \( S \), can be written as

\[
\vec{G} = \int_s \vec{n} \times \vec{r}' \, p \, ds
\]  

(14)

where

\[ \vec{r}' = \vec{r} - \vec{R} \]

and \( \vec{R} \) is the position vector of the center of mass of the fluid within \( S \), and \( \vec{n} \) as before, is the unit normal vector to the contour \( s \) of \( S \) directed away from the solid cylinder.

We now calculate \( \vec{G} \) directly considering moments of individual fluid particles about the center of mass. The net moment of all the forces acting on the fluid is given by the integral over all contributions \( \rho \vec{r} \times \left( \frac{d\vec{u}}{dt} \right) \, dS \) from fluid particles with mass \( \rho \, dS \) located at \( \vec{r} \). Therefore this moment is given by

\[
\int \rho \vec{r} \times \left( \frac{d\vec{u}}{dt} \right) \, dS.
\]

(15)

We observe that in the integral (15) the contributions of the internal action between different fluid particles may be neglected since these internal actions can be expressed as pairs of equal and opposite collinear forces and these forces have zero net moment about the center of mass. The integral (15) thus is identical with the moment of the external forces given by \( \vec{G} \) in (14). In view of (1), we get

\[
\vec{G} = \int \rho \vec{r} \times \left( \frac{d\vec{u}}{dt} + 2\vec{\omega} \times \vec{u} - \omega^2 \vec{r} \right) \, dS
\]

(16)

If \( \vec{G}' \) is the moment of the same flow but in the absence of rotation (\( \omega = 0 \)), equation (16) shows that

\[
\vec{G} - \vec{G}' = \int \rho \vec{r} \times (2\vec{\omega} \times \vec{u} - \omega^2 \vec{r}) \, dS = \int \rho (2 \vec{r} \times \vec{u} \omega - \omega^2 \vec{r} - \vec{r}) \, dS
\]

(17)
as $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{R}$ are in a plane (xy-plane) perpendicular to $\mathbf{\omega}$ and hence $\mathbf{\omega} \cdot \mathbf{r}' = 0$. Defining $\mathbf{\bar{u}} = \mathbf{\bar{U}} + \mathbf{\bar{u}}'$, where $\mathbf{\bar{u}}'$ is the velocity of fluid relative to the center of mass, and noting, in view of definition of $\mathbf{r}$ and $\mathbf{R}$, that $\int \rho \mathbf{r} \times dS = \int \rho \mathbf{r}_dS = 0$, we obtain

$$
\int \rho (\mathbf{r} \cdot \mathbf{\bar{u}}) dS = \int \rho (\mathbf{r} \cdot \mathbf{\bar{u}} + \mathbf{r} \cdot \mathbf{\bar{U}}) dS = \int \rho \mathbf{r} \cdot \mathbf{\bar{u}}' dS = \int \rho \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} dS;
$$

(18)

where we have used $\int \rho \mathbf{r} \cdot \mathbf{\bar{U}} dS = 0$, since $\int \mathbf{r}' dS = 0$.

Using (9), we get from (18) the relation

$$
\int \rho \mathbf{r} \cdot \mathbf{\bar{u}} dS = \frac{1}{2} \frac{d}{dt} \int \rho \mathbf{r}'^2 dS;
$$

(19)

Equation (17), in view of (18) and (19), finally leads to the result

$$
\mathbf{G} - \mathbf{G}' = \mathbf{\omega} \frac{d}{dt} \int \rho \mathbf{r}'^2 dS.
$$

(20)

This is the extra moment exerted on a cylinder moving through an incompressible ideal fluid. In an incompressible fluid of constant density, as considered by Taylor [5], the integral in (20) is a constant, being the moment of inertia of a solid body of density $\rho$ about it center of mass, and hence $\mathbf{G} - \mathbf{G}'$.

3 Concluding remarks

The method used here to derive the general expressions for the extra force and moment on cylinders in a rotating ideal fluid is equivalent to the method of images applied successfully to other problems in Fluid Mechanics ([3], [4]). In discussing the two-dimensional flow past a solid cylinder by the method of images, the cylinder is assumed to be removed and the fluid flow to be extended to the region previously occupied by the cylinder. The main propose of the derivation here is to examine the possibility of extending the results obtained earlier by Taylor to the case of compressible fluid and to make these better accessible to the readers by using a different method. Taylor himself noted about his work: “... that it is hoped that the interest which it gives to the mathematical work will serve to extenuate, to a certain extent, the clumsiness of the methods employed.”

We find that Taylor’s results cannot be extended to the compressible fluid.

References


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