Functions of Slow Increase Generalization of the Logarithmic Integral

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Abstract

We generalize the well-known formula.

\[ Li(x) = \int_2^x \frac{1}{\log t} \, dt = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \cdots + \frac{(m - 1)!x}{\log^m x} + o\left(\frac{x}{\log^m x}\right). \]

We prove a similar formula for \( \int_b^x f(t) \, dt \) if \( f(x) \) is a function of slow increase and certain conditions are fulfilled.

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1 Introduction

First, we recall the definition of function of slow increase given in [1].

Definition 1.1 Let \( f(x) \) be a function defined on the interval \([a, \infty)\) such that \( f(x) > 0, \lim_{x \to \infty} f(x) = \infty \) and with continuous derivative \( f'(x) > 0 \). The function \( f(x) \) is of slow increase if and only if the following condition holds.

\[ \lim_{x \to \infty} \frac{xf'(x)}{f(x)} = 0. \]  

Typical functions of slow increase are \( f(x) = \log x, f(x) = \log^2 x, f(x) = \log \log x \). We have the following theorem.

Theorem 1.2 Let us consider a function \( f(x) \) of slow increase defined on the interval \([a, \infty)\) (\( a > 0 \)). Suppose that (see (1))

\[ \frac{xf'(x)}{f(x)} = \frac{1}{g(x)}. \]
Where \( g(x) \) is a function of slow increase on the interval \([a, \infty)\). Then \( f(x) \) has a continuous second derivative \( f''(x) \) and

\[
\lim_{x \to \infty} \frac{x f''(x)}{f'(x)} = -1. \tag{3}
\]

Proof. Equation (2) gives

\[
f'(x) = \frac{f(x)}{xg(x)}. \tag{4}
\]

Equation (4) gives

\[
f''(x) = \frac{x f'(x)g(x) - f(x)g(x) - xf(x)g'(x)}{x^2 g(x)^2}. \tag{5}
\]

Therefore \( f''(x) \) is continuous on \([a, \infty)\). Finally, equations (4) and (5) give

\[
\lim_{x \to \infty} \frac{x f''(x)}{f'(x)} = \lim_{x \to \infty} \left( \frac{xf'(x)}{f(x)} - 1 - \frac{xg'(x)}{g(x)} \right) = -1.
\]

Since \( f(x) \) and \( g(x) \) are functions of slow increase. The theorem is proved.

2 Generalization of the Logarithmic Integral

It is well-known the formula \((m \geq 1)\).

\[
Li(x) = \int_2^x \frac{1}{\log t} \, dt = \frac{x}{\log x} + \frac{1! x}{\log^2 x} + \cdots + \frac{(m - 1)! x}{\log^m x} + o \left( \frac{x}{\log^m x} \right)
\]

\[
= \sum_{k=1}^{m} \frac{(k - 1)! x}{\log^k x} + o \left( \frac{x}{\log^m x} \right). \tag{6}
\]

In this section we generalize this formula if certain conditions are fulfilled.

We have the following theorem.

**Theorem 2.1** Let us consider a function \( f(x) \) of slow increase defined on the interval \([a, \infty) \) \((a > 0)\). Suppose that (see (2))

\[
xf'(x) = \frac{1}{g(x)} \tag{7}
\]

where \( g(x) \) is a function of slow increase on the interval \([a, \infty)\) and suppose that (see (3) and (1))

\[
\frac{xf''(x)}{f'(x)} = -1 + o \left( \frac{xf'(x)}{f(x)} \right)^{m-1}, \tag{8}
\]
where \( m \) is a certain positive integer. Then the following formula holds \((b \geq a)\).

\[
\int_b^x \frac{1}{f(t)} \, dt = \frac{x}{f(x)} + \frac{1}{f(x)} \frac{x^2 f'(x)}{2} + \cdots + \frac{(m-1)! x^m f'(x)^{m-1}}{m \cdot f(x)^m} + \frac{x^m f'(x)^{m-1}}{f(x)^m}.
\]

(9)

**Remark 2.2** Note that in equation (9) we have (see the proof of Theorem 7 in [1])

\[
\lim_{x \to \infty} \int_b^x \frac{1}{f(t)} \, dt = \infty.
\]

**Remark 2.3** Note that in equation (9) we have (see Theorem 2 and Theorem 4 in [1])

\[
\lim_{x \to \infty} \frac{(k-1)! x^k (f'(x))^{k-1}}{f(x)^k} = \lim_{x \to \infty} (k-1)! \frac{x}{f(x)} \left( \frac{x f'(x)}{f(x)} \right)^{k-1} = \lim_{x \to \infty} (k-1)! \frac{x}{f(x)} g(x)^{k-1} = \infty \quad (k = 1, \ldots, m).
\]

Note also that in equation (9) we have (see (1))

\[
\lim_{x \to \infty} \frac{(k+1)! x^{k+1} (f'(x))^{k+1}}{f(x)^{k+1}} = \lim_{x \to \infty} k \frac{x f'(x)}{f(x)} = 0 \quad (k = 1, \ldots, m - 1).
\]

**Remark 2.4** Note that if \( f(t) = \log t \) the conditions of Theorem 2.1 are fulfilled (in particular (8) is true for all positive integer \( m \)). In this case (9) becomes (6). Note also that in Theorem 2.1, \( f''(x) \) is continuous (see Theorem 1.2).

Proof. We have (see Theorem 7 in [1])

\[
\int_b^x \frac{1}{f(t)} \, dt \sim \frac{x}{f(x)}.
\]

That is,

\[
\int_b^x \frac{1}{f(t)} \, dt = \frac{x}{f(x)} + o \left( \frac{x}{f(x)} \right).
\]

Therefore (9) is true if \( m = 1 \).
Suppose that $m \geq 2$. We have (integration by parts)

$$\int_b^x \frac{1}{f(t)} \, dt = \frac{x}{f(x)} + O(1) + \int_b^x \frac{tf'(t)}{f(t)^2} \, dt. \tag{10}$$

On the other hand we have (integration by parts)

$$\int_b^x \frac{k!t^k(f'(t))^k}{f(t)^{k+1}} \, dt = \frac{k!x^{k+1}(f'(x))^k}{f(x)^{k+1}} + O(1) - \int_b^x \frac{t}{f(t)^{k+1}} \left( \frac{k!t^k(f'(t))^k}{f(t)^{k+1}} \right) \, dt$$

$$= \frac{k!x^{k+1}(f'(x))^k}{f(x)^{k+1}} + O(1) + \int_b^x \frac{(k+1)!x^{k+1}(f'(t))^{k+1}}{f(t)^{k+2}} \, dt + \int_b^x \left( -\frac{k!kt^k(f'(t))^k}{f(t)^{k+1}} \right) f''(t) \, dt \quad (k = 1, \ldots, m - 1). \tag{11}$$

Equation (10) and equation (11) applied repeatedly give us,

$$\int_b^x \frac{1}{f(t)} \, dt = \sum_{k=1}^{m} \frac{(k-1)!x^k(f'(x))^{k-1}}{f(x)^k} + O(1) + \int_b^x \left( \frac{m!t^m(f'(t))^m}{f(t)^{m+1}} \right) \, dt$$

$$- \sum_{k=1}^{m-1} \left( \frac{k!kt^k(f'(t))^k}{f(t)^{k+1}} + \frac{k!kt^k(f'(t))^{k+1}t}{f(t)^{k+1}} \right) \, dt$$

$$= \sum_{k=1}^{m} \frac{(k-1)!x^k(f'(x))^{k-1}}{f(x)^k} + O(1) + \int_b^x \left( \frac{m!t^m(f'(t))^m}{f(t)^{m+1}} \right) \, dt$$

$$- \sum_{k=1}^{m-1} \left( \frac{k!kt^k(f'(t))^k}{f(t)^{k+1}} + o \left( \frac{t}{f(t)} \right)^{m-1} \right) \, dt$$

$$= \sum_{k=1}^{m} \frac{(k-1)!x^k(f'(x))^{k-1}}{f(x)^k} + O(1) + \int_b^x h(t) \frac{m!t^m(f'(t))^m}{f(t)^{m+1}} \, dt.$$

Where $h(t) \to 1$ (note that $h(t)$ is continuous on $[a, \infty)$). That is,

$$\int_b^x \frac{1}{f(t)} \, dt = \sum_{k=1}^{m} \frac{(k-1)!x^k(f'(x))^{k-1}}{f(x)^k} + O(1) + \int_b^x h(t) \frac{m!t^m(f'(t))^m}{f(t)^{m+1}} \, dt. \tag{12}$$

We have

$$h(t) \frac{m!t^m(f'(t))^m}{f(t)^{m+1}} = m! \frac{h(t)}{g(t)^m} \frac{1}{f(t)}. $$
Where \( g(t)^m f(t) \) is a function of slow increase (see Theorem 2 in [1]). Consequently (see the proof of Theorem 7 in [1])

\[
\lim_{x \to \infty} \int_b^x h(t) \frac{m!t^m (f'(t))^m}{f(t)^{m+1}} \, dt = \infty. \tag{13}
\]

We have

\[
\lim_{x \to \infty} \frac{\int_b^x h(t) \frac{m!t^m (f'(t))^m}{f(t)^{m+1}} \, dt}{(m-1)!x^m (f'(x))^{m-1}} = \lim_{x \to \infty} m \frac{x f'(x) \int_b^x h(t) \frac{m!t^m (f'(t))^m}{f(t)^{m+1}} \, dt}{f(x)^m} \tag{14}
\]

Using the L’Hospital’s rule we obtain (see (13), (1) and (3))

\[
\lim_{x \to \infty} \frac{\int_b^x h(t) \frac{m!t^m (f'(t))^m}{f(t)^{m+1}} \, dt}{(m-1)!x^m (f'(x))^{m-1}} = \lim_{x \to \infty} \frac{h(x) x^{m+1} (f'(x))^m}{f(x)^{m+1}} = \lim_{x \to \infty} \frac{d}{dx} \left( \frac{x^{m+1} (f'(x))^m}{f(x)^{m+1}} \right) = 1. \tag{15}
\]

Equations (14), (15) and (1) give

\[
\lim_{x \to \infty} \frac{\int_b^x h(t) \frac{m!t^m (f'(t))^m}{f(t)^{m+1}} \, dt}{(m-1)!x^m (f'(x))^{m-1}} = 0. \tag{16}
\]

Finally (12) and (16) give

\[
\int_b^x \frac{1}{f(t)} \, dt = \sum_{k=1}^{m} \frac{(k-1)!x^k (f'(x))^{k-1}}{f(x)^k} + o \left( \frac{x^m (f'(x))^{m-1}}{f(x)^m} \right).
\]

That is (9). The theorem is proved.

**Theorem 2.5** Suppose that \( g(x) \) is a function of slow increase defined on the interval \([a, \infty)\) \((a > 0)\) such that

\[
\lim_{x \to \infty} \int_b^x \frac{1}{t g(t)} \, dt = \infty \quad (b \geq a). \tag{17}
\]

Then there exists an unique function \( f(x) \) of slow increase defined on the interval \([a, \infty)\) with continuous second derivative such that \( f(b) = c > 0 \) and such that

\[
\frac{x f'(x)}{f(x)} = \frac{1}{g(x)}. \tag{18}
\]

This function is

\[
f(x) = c \exp \left( \int_b^x \frac{1}{t g(t)} \, dt \right). \tag{19}
\]
Proof. Equation (18) can be written in the form
\[ f'(x) - \frac{1}{xg(x)} f(x) = 0. \]
This is a homogeneous linear differential equation of first order well-known. All the solutions of this differential equation are
\[ f(x) = f(b) \exp\left( \int_b^x \frac{1}{tg(t)} \, dt \right). \]
Therefore since \( f(b) = c \) our solution is
\[ f(x) = c \exp\left( \int_b^x \frac{1}{tg(t)} \, dt \right). \]
Clearly (see Definition 1.1 and (17)) this function is of slow increase on the interval \([a, \infty)\). Its first derivative is
\[ f'(x) = \frac{c \exp\left( \int_b^x \frac{1}{tg(t)} \, dt \right)}{xg(x)}. \] (20)
Its second derivative is
\[ f''(x) = \frac{c \exp\left( \int_b^x \frac{1}{tg(t)} \, dt \right)}{x^2g(x)^2} \left( 1 - g(x) - xg'(x) \right). \] (21)
The theorem is proved.

**Theorem 2.6** If the condition (see (8))
\[ \frac{xf''(x)}{f'(x)} = -1 + o\left( \frac{xf'(x)}{f(x)} \right)^{m-1} = -1 + o\left( \frac{1}{g(x)} \right)^{m-1} \] (22)
is fulfilled \((m \geq 2 \text{ is a certain positive integer})\) then \(g(x) \sim \log x\).

Proof. Equations (20) and (21) give
\[ \frac{xf''(x)}{f'(x)} = -1 + \frac{1 - xg'(x)}{g(x)} = -1 + (1 - xg'(x))g(x)^{m-2} \left( \frac{1}{g(x)} \right)^{m-1}. \] (23)
Equations (22) and (23) give \((1 - xg'(x))g(x)^{m-2} = o(1)\), that is \((1 - xg'(x)) = o(1)\). Hence
\[ g'(x) = \frac{h(x)}{x}. \] (24)
Where \( h(x) \to 1 \) (note that \( h(x) \) is continuous on \([a, \infty)\)). Therefore (L’Hospital’s rule)

\[
\lim_{x \to \infty} \frac{g(x)}{\log x} = \lim_{x \to \infty} \frac{h(x)}{x} = \lim_{x \to \infty} h(x) = 1.
\]

The theorem is proved.

Finally, we give an example where condition (22) (that is, condition (8)) is fulfilled. Let us consider the following function \( f(x) \).

\[
f(x) = c_1 \exp \left( \int_b^x \frac{1}{t g(t)} \, dt \right) \quad (c_1 > 0) \quad (b \geq a)
\]

where

\[
g(x) = c_2 + \int_a^x \left( \frac{1}{t} - \frac{1}{t (\log t)^{m-1}} \right) \, dt \sim \log x \quad (c_2 > 0) \quad (a > e) \quad (m \geq 2).
\]

Clearly \( g(x) \) is a function of slow increase and \( f(x) \) is also a function of slow increase since (Theorem 2.5)

\[
\frac{xf'(x)}{f(x)} = \frac{1}{g(x)}.
\]

We have to prove that (see (23)) \((1 - xg'(x))g(x)^{m-2} \to 0\). Now, we have

\[
(1 - xg'(x))g(x)^{m-2} = \left( 1 - x \left( \frac{1}{x} - \frac{1}{x (\log x)^{m-1}} \right) \right) h(x)^{m-2} \log^{m-2} x
\]

\[
= \frac{h(x)^{m-2}}{\log x} \to 0 \quad (h(x) \to 1).
\]

As we desired.

An example (different of \( \log x \)) where condition (22) (that is, condition (8)) is fulfilled for all \( m \geq 2 \) is the following.

\[
f(x) = c_1 \exp \left( \int_b^x \frac{1}{t g(t)} \, dt \right) \quad (c_1 > 0) \quad (b \geq a)
\]

where

\[
g(x) = c_2 + \int_a^x \left( \frac{1}{t} - \frac{1}{t^{1+\alpha}} \right) \, dt \sim \log x \quad (c_2 > 0) \quad (a > 1) \quad (\alpha > 0).
\]

References


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