Operations in Generalized Fuzzy Topological Spaces

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Abstract

Generalized fuzzy topological space was introduced and studied by Palani Chetty in 2008. The aim of this paper is to define operations on the family of fuzzy sets of a set and discuss its properties.

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1. Introduction and Preliminaries

Let $X$ be a nonempty set and $\mathcal{F} = \{\lambda : \lambda : X \to [0, 1]\}$ be the family of all fuzzy sets defined on $X$. Let $\gamma : \mathcal{F} \to \mathcal{F}$ be a function such that $\lambda \leq \mu$ implies that $\gamma(\lambda) \leq \gamma(\mu)$ for every $\lambda, \mu \in \mathcal{F}$. That is, each $\gamma$ is a monotonic function defined on $\mathcal{F}$. We will denote the collection of all monotonic functions defined on $\mathcal{F}$ by $\Gamma(\mathcal{F})$ or simply $\Gamma$. Let $\gamma \in \Gamma$. A fuzzy set $\lambda \in \mathcal{F}$ is said to be $\gamma$—fuzzy open [3] if $\lambda \leq \gamma(\lambda)$. Clearly, $\bar{0}$, the null fuzzy set is $\gamma$—fuzzy open. In [3], it is established that the arbitrary union of $\gamma$—fuzzy open sets is again a $\gamma$—fuzzy open set. A subfamily $\mathcal{G}$ of $\mathcal{F}$ is called a generalized fuzzy topology(GFT) [3] if $\bar{0} \in \mathcal{G}$ and $\lor \{\lambda_\alpha \mid \alpha \in \Delta\} \in \mathcal{G}$ whenever $\lambda_\alpha \in \mathcal{G}$ for every $\alpha \in \Delta$. If $\gamma \in \Gamma$, it follows that $\mathcal{A}$, the family of all $\gamma$—fuzzy open sets is a generalized
fuzzy topology. For $\lambda \in \mathcal{F}$, the $\gamma-$interior of $\lambda$, denoted by $i_\gamma(\lambda)$, is given
by $i_\gamma(\lambda) = \{\nu \in \mathcal{A} \mid \nu \leq \lambda\}$. Moreover, in [1], it is established that for all $\lambda \in \mathcal{F}$,
$i_\gamma(\lambda) \subseteq \lambda$, $i_\gamma(\lambda) = i_\gamma(\lambda)$ and $\lambda \in \mathcal{A}$ if and only if $\lambda = i_\gamma(\lambda)$. A fuzzy
set $\lambda \in \mathcal{F}$ is said to be a $\gamma-$fuzzy closed set if $\overline{\lambda} = \lambda$ is a $\gamma-$fuzzy open set. The
intersection of all $\gamma-$fuzzy closed sets containing $\lambda \in \mathcal{F}$ is called the $\gamma-$closure of $\lambda$. It is denoted by $c_\gamma(\lambda)$
and is given by $c_\gamma(\lambda) = \{\mu \mid \overline{\mu} - \mu \in \mathcal{A}, \lambda \leq \mu\}$. In [1], it is established that $c_\gamma(\lambda) = \overline{\lambda} - i_\gamma(\overline{\lambda} - \lambda)$ for all $\lambda \in \mathcal{F}$. A fuzzy point [4]
$x_\alpha$, with support $x \in X$ and value $0 < \alpha \leq 1$ is defined by $x_\alpha(y) = \alpha$, if $y = x$ and $x_\alpha(y) = 0$, if $y \neq x$. Again, for $\lambda \in \mathcal{F}$, we say that $x_\alpha \in \lambda$ if $\alpha \leq \lambda(x)$.
Two fuzzy sets $\lambda$ and $\beta$ are said to be quasi-coincident [4], denoted by $\lambda \beta$, if there exists $x \in X$ such that $\lambda(x) + \beta(x) > 1$ [4]. Two fuzzy sets $\lambda$ and $\beta$ are not quasi-coincident denoted by $\lambda \beta$, if $\lambda(x) + \beta(x) \leq 1$ for all $x \in X$. Clearly, $\lambda$ is a $\gamma-$fuzzy open set containing a point $x_\alpha$ if and only if $x_\alpha \lambda$, and $\lambda \leq \beta$ if and only if $\lambda \beta(\overline{\beta} - \beta)$. For definitions not given here, refer [2].

2. Enlarging and quasi-Enlarging operations

Let $X$ be a nonempty set and $\gamma \in \Gamma$. Let us agree in calling operation, any element of $\Gamma$. An operation $\gamma \in \Gamma$ is said to be enlarging if $\lambda \leq \gamma(\lambda)$ for every $\lambda \in \mathcal{F}$. If $\mathcal{B} \subset \mathcal{F}$, then $\gamma \in \Gamma$ is said to be $\mathcal{B}-$enlarging if $\lambda \leq \gamma(\lambda)$ for every $\lambda \in \mathcal{B}$. We will denote the family of all enlarging operations by $\Gamma_e$ and the family of all $\mathcal{B}-$enlarging operations by $\Gamma_{\mathcal{B}}$. The easy proof of the following Theorem 2.1 is omitted.

**Theorem 2.1.** Let $X$ be a nonempty set and $\mathcal{F}$ be the family of all fuzzy sets defined on $X$. If $\mathcal{C} \subset \mathcal{B} \subset \mathcal{F}$, then $\Gamma_{\mathcal{B}} \subset \Gamma_{\mathcal{C}}$. $\Gamma_e = \Gamma_{\mathcal{B}}$, if $\mathcal{B} = \mathcal{F}$.

An operation $\gamma \in \Gamma$, is said to be quasi-enlarging (QE) if $\gamma(\lambda) \leq \gamma(\lambda \land \gamma(\lambda))$ for every $\lambda \in \mathcal{F}$. An operation $\gamma \in \Gamma$, is said to be weakly quasi-enlarging (WQE) if $\lambda \land \gamma(\lambda) \leq \gamma(\lambda \land \gamma(\lambda))$ for every $\lambda \in \mathcal{F}$. If $\gamma \in \Gamma_e$, then $\lambda \land \gamma(\lambda) = \lambda$ for every $\lambda \in \mathcal{F}$ and so $\gamma$ is quasi-enlarging. If $\gamma$ is defined by $\gamma(\lambda) = \beta$ for every $\lambda \in \mathcal{F}$, then also $\gamma$ is quasi-enlarging. If $\gamma \in \Gamma$ is quasi-enlarging, then it is weakly quasi-enlarging, since $\lambda \land \gamma(\lambda) \leq \gamma(\lambda) \leq \gamma(\lambda \land \gamma(\lambda))$. The following Example 2.2 shows that a weakly quasi-enlarging operation need not be a quasi-enlarging operation.

**Example 2.2.** Let $X = \{x, y, z\}$. Define $\gamma : \mathcal{F} \to \mathcal{F}$, by $\gamma(\lambda) = \overline{\lambda}$, if $\lambda = \overline{0}$; $\gamma(\lambda) = x_\{y\}$, if $\lambda \leq x_\{x\}$; $\gamma(\lambda) = x_\{z\}$, if $\lambda \leq x_\{z\}$ and $\gamma(\lambda) = \overline{1}$ if otherwise. Then, $\lambda \land \gamma(\lambda) = \overline{0}$, if $\lambda = \overline{0}$; $\lambda \land \gamma(\lambda) = \overline{0}$, if $\lambda \leq x_\{z\}$; $\lambda \land \gamma(\lambda) \leq x_\{z\}$, if $\lambda \leq x_\{z\}$ and $\lambda \land \gamma(\lambda) = \lambda$ if otherwise. Therefore, $\gamma(\lambda \land \gamma(\lambda)) = \overline{0}$, if $\lambda = \overline{0}$; $\gamma(\lambda \land \gamma(\lambda)) = \overline{0}$, if $\lambda \leq x_\{z\}$; $\gamma(\lambda \land \gamma(\lambda)) = x_\{z\}$, if $\lambda \leq x_\{z\}$ and $\lambda \land \gamma(\lambda) = \overline{1}$, if otherwise and so it follows that $\gamma$ is a weakly quasi-enlarging operator. If $\lambda = x_\{x\}$, then $\gamma(\lambda) = x_\{y\}$ but $\gamma(\lambda \land \gamma(\lambda)) = \gamma(\overline{0}) = \overline{0}$ and so $\gamma$ is not a quasi-enlarging operator.

If $\gamma_1, \gamma_2 \in \Gamma$, then the composition $\gamma_1 \circ \gamma_2$ of the two operations $\gamma_1$ and $\gamma_2$ is again an operation and we write $\gamma_1 \gamma_2$ instead of $\gamma_1 \circ \gamma_2$. The following Theorem 2.3 shows that the composition of enlarging operators is again an
enlarging operator and Theorem 2.5 below gives a property of quasi-enlarging operators.

**Theorem 2.3.** Let $X$ be a nonempty set and $\mathcal{F}$ be the family of all fuzzy sets defined on $X$. If $\mathcal{B} \subset \mathcal{F}$, and $\gamma_1$ and $\gamma_2$ are $\mathcal{B}$-enlarging, then $\gamma_1 \gamma_2$ is also $\mathcal{B}$-enlarging.

**Proof.** Suppose $\lambda \in \mathcal{B}$. Then $\lambda \leq \gamma_1(\lambda)$ and $\lambda \leq \gamma_2(\lambda)$. Now, $\lambda \leq \gamma_1(\gamma_2(\lambda))$, since $\gamma_1 \in \Gamma$. Therefore, $\gamma_1 \gamma_2$ is $\mathcal{B}$-enlarging.

**Corollary 2.4.** If $\gamma_1$, $\gamma_2 \in \Gamma_e$, then $\gamma_1 \gamma_2 \in \Gamma_e$.

**Theorem 2.5.** Let $X$ be a nonempty set, $\mathcal{F}$ be the family of all fuzzy sets defined on $X$ and $\mathcal{B} \subset \mathcal{F}$. If $\gamma \in \Gamma$ is quasi-enlarging, $\{\gamma(\lambda) \mid \lambda \in \mathcal{F}\} \subset \mathcal{B}$ and $\mu \in \Gamma_\mathcal{B}$, then $\mu \gamma$ is quasi-enlarging.

**Proof.** Let $\lambda \in \mathcal{F}$. Since $\gamma$ is quasi-enlarging, $\gamma(\lambda) \leq \gamma(\lambda \wedge \gamma(\lambda))$. Since $\gamma(\lambda) \in \mathcal{B}$ and $\mu \in \Gamma_\mathcal{B}$, $\gamma(\lambda) \leq \mu(\gamma(\lambda))$ and so $\gamma(\lambda) \leq \gamma(\lambda \wedge \mu \gamma(\lambda))$. Therefore, $\mu \gamma(\lambda) \leq \mu \gamma(\lambda \wedge \mu \gamma(\lambda))$. Hence $\mu \gamma$ is quasi-enlarging.

**Theorem 2.6.** Let $X$ be a nonempty set and $\gamma \in \Gamma$. Then $i_\gamma$ is quasi-enlarging and $c_\gamma$ is enlarging.

**Proof.** If $\lambda \in \mathcal{F}$, then $i_\gamma(\lambda) = i_\gamma i_\gamma(\lambda) = i_\gamma(\lambda \wedge i_\gamma(\lambda))$, since $i_\gamma(\lambda) \leq \lambda$. So $i_\gamma$ is quasi-enlarging. Again, $i_\gamma(1 - \lambda) \leq 1 - \lambda$ and so $\lambda = 1 - (1 - \lambda) \leq 1 - i_\gamma(1 - \lambda) = c_\gamma(\lambda)$. Therefore, $c_\gamma$ is enlarging.

**Theorem 2.7.** Let $X$ be a nonempty set, $\gamma \in \Gamma$ and $\mathcal{A}$ be the family of all $\gamma$-fuzzy open sets. If $\mu \in \Gamma$, such that $i_\gamma \mu$ is quasi-enlarging and $\kappa \in \Gamma_\mathcal{A}$, then $\kappa i_\gamma \mu$ is quasi-enlarging.

**Proof.** If $\lambda \in \mathcal{F}$, then $i_\gamma \mu(\lambda) \in \mathcal{A}$. By Theorem 2.5, it follows that $\kappa i_\gamma \mu$ is quasi-enlarging.

**Corollary 2.8.** Let $X$ be a nonempty set, $\gamma \in \Gamma$ and $\mathcal{A}$ be the family of all $\gamma$-fuzzy open sets. If $\kappa \in \Gamma_\mathcal{A}$, then $\kappa i_\gamma$ is quasi-enlarging.

**Proof.** If $\mu : \mathcal{F} \rightarrow \mathcal{F}$ is the identity operator, then $i_\gamma \mu = i_\gamma$ is quasi-enlarging and so the proof follows from Theorem 2.7.

Let $\{\gamma_\iota \in \Gamma \mid \iota \in \Delta \neq \emptyset\}$ be a family of operations. Define $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ by $\varphi(\lambda) = \vee \{\gamma_\iota(\lambda) \mid \iota \in \Delta\}$ for every $\lambda \in \mathcal{F}$. The following Theorem 2.9 gives some properties of $\varphi$.

**Theorem 2.9.** Let $X$ be a nonempty set. Let $\{\gamma_\iota \in \Gamma \mid \iota \in \Delta \neq \emptyset\}$ be a family of operations. Define $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ by $\varphi(\lambda) = \vee \{\gamma_\iota(\lambda) \mid \iota \in \Delta\}$ for every $\lambda \in \mathcal{F}$. Then the following hold.

(a) $\varphi \in \Gamma$.
(b) If each $\gamma_\iota$ is $\mathcal{B}$-enlarging, then so is $\varphi$.
(c) If each $\gamma_\iota$ is quasi-enlarging, then so is $\varphi$.
(d) If each $\gamma_\iota$ is weakly quasi-enlarging, then so is $\varphi$.

**Proof.** (a) If $\lambda \leq \nu$, then $\gamma_\iota(\lambda) \leq \gamma_\iota(\nu)$ and so $\varphi(\lambda) = \vee \{\gamma_\iota(\lambda) \mid \iota \in \Delta\} \leq \vee \{\gamma_\iota(\nu) \mid \iota \in \Delta\} = \varphi(\nu)$. Therefore, $\varphi \in \Gamma$.

(b) Let $\lambda \in \mathcal{B}$. Then, by hypothesis, $\lambda \leq \gamma_\iota(\lambda)$ for every $\iota \in \Delta \neq \emptyset$. Therefore, $\lambda \leq \vee \gamma_\iota(\lambda) = \varphi(\lambda)$ and so $\varphi$ is $\mathcal{B}$-enlarging.
(c) Suppose each $\gamma_i$ is quasi-enlarging. Then for $\lambda \in \mathcal{F}$, $\varphi(\lambda) = \vee \gamma_i(\lambda) \leq \vee \gamma_i(\lambda \land \gamma_i(\lambda)) \leq \vee \gamma_i(\lambda \land \varphi(\lambda)) = \varphi(\lambda \land \varphi(\lambda))$ and so $\varphi$ is quasi-enlarging.

(d) For $\lambda \in \mathcal{F}$, $\lambda \land \varphi(\lambda) = \lambda \land (\vee \gamma_i(\lambda)) = \vee (\lambda \land \gamma_i(\lambda)) \leq \vee \gamma_i(\lambda \land \gamma_i(\lambda)) \leq \vee \gamma_i(\lambda \land \varphi(\lambda)) = \varphi(\lambda \land \varphi(\lambda))$. Therefore, $\varphi$ is weakly quasi-enlarging.

**Definition 2.10.** Let $X$ be a nonempty set and $\mathcal{A} \subset \mathcal{F}$. We say that an operation $\gamma \in \Gamma$ is $\mathcal{A}$-friendly, if $\nu \land \gamma(\lambda) \leq \gamma(\nu \land \lambda)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \mathcal{A}$.

The following Example 2.11 gives examples of $\mathcal{A}$-friendly operators. It is clear that if $\gamma$ is $\mathcal{A}$-friendly and $\mathcal{B} \subset \mathcal{A}$, then $\gamma$ is a $\mathcal{B}$-friendly operator. Theorem 2.12 below shows that the composition of friendly operators is again a friendly operator. Theorem 2.13 shows that arbitrary union of friendly operators is again a friendly operator.

**Example 2.11.** (a) If $\gamma : \mathcal{F} \to \mathcal{F}$ is defined by $\gamma(\lambda) = \theta$ for every $\lambda \in \mathcal{F}$ for some $\theta \in \mathcal{F}$, then $\gamma$ is $\mathcal{A}$-friendly for every $\mathcal{A} \subset \mathcal{F}$.

(b) In any fuzzy topological space $(X, \tau)$, the fuzzy interior and closure operators $i_\tau$ and $c_\tau$ are $\tau$-friendly. That is, the following hold.

(i) $i_\tau(\lambda) \land \nu \leq i_\tau(\lambda \land \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \tau$.

(ii) $c_\tau(\lambda) \lor \nu \leq c_\tau(\lambda \lor \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \tau$.

**Theorem 2.12.** Let $X$ be a nonempty set, $\gamma, \gamma_1 \in \Gamma$ and $\mathcal{A} \subset \mathcal{F}$. If $\gamma$ and $\gamma_1$ are $\mathcal{A}$-friendly operators, then so is $\gamma_1 \gamma$.

**Proof.** Suppose $\mathcal{A} \subset \mathcal{F}$ such that $\gamma$ and $\gamma_1$ are $\mathcal{A}$-friendly. Then, $\gamma(\lambda) \lor \nu \leq \gamma(\lambda \lor \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \mathcal{A}$, and $\gamma_1(\lambda) \lor \nu \leq \gamma_1(\lambda \lor \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \mathcal{A}$. Replacing $\lambda$ by $\gamma(\lambda)$ in the second inequality, we get $\gamma_1 \gamma(\lambda) \lor \nu \leq \gamma_1(\gamma(\lambda) \lor \nu) \leq \gamma_1^1(\lambda \lor \nu)$. Therefore, $\gamma_1 \gamma$ is an $\mathcal{A}$-friendly operator.

**Theorem 2.13.** Let $X$ be a nonempty set, $\mathcal{A} \subset \mathcal{F}$ and $\gamma_i$ is $\mathcal{A}$-friendly for every $i \in \Delta$. Then $\varphi = \vee \gamma_i$ is $\mathcal{A}$-friendly.

**Proof.** If $\lambda \in \mathcal{F}$, then for $\nu \in \mathcal{A}$, $\varphi(\lambda) \lor \nu = (\vee \gamma_i)(\lambda) \lor \nu = \vee(\gamma_i(\lambda) \lor \nu) \leq \vee \gamma_i(\lambda \lor \nu) = \varphi(\lambda \lor \nu)$. Therefore, $\varphi$ is an $\mathcal{A}$-friendly operator.

Using friendly operators, next we construct quasi-enlarging operators using a generalized fuzzy topology (GFT). Let $\mu \subset \mathcal{F}$ be arbitrary. For $\lambda \in \mathcal{F}$, define $i_\mu(\lambda) = \vee \{\beta \in \mu \mid \beta \leq \lambda\}$ and $i_\mu(\lambda) = 0$, if no such $\beta \in \mu$ exists. Let $\mu' = \{1 - \lambda \mid \lambda \in \mu\}$. Define $c_\mu(\lambda) = \vee \{\beta \in \mu \mid \beta \leq \lambda\}$ and $c_\mu(\lambda) = 1$, if no such $\beta \in \mu'$ exists. If $\mu$ is the family of all $\gamma$-open sets, then $c_\gamma = c_\mu$ and $i_\gamma = i_\mu$.

**Theorem 2.14.** Let $\mu \subset \mathcal{F}$ be a GFT. If $\gamma \in \Gamma$, is $\mu$-friendly, then $i_\mu \gamma$ is quasi-enlarging.

**Proof.** If $\xi \in \mathcal{F}$, then $i_\mu \gamma(\xi) = (\gamma(\xi) \land i_\mu \gamma(\xi))$. Since $\gamma$ is $\mu$-friendly, $\gamma(\xi) \land i_\mu \gamma(\xi) \leq \gamma(\xi \land i_\mu \gamma(\xi))$. Therefore, $i_\mu \gamma(\xi) = i_\mu i_\mu \gamma(\xi) \leq i_\mu \gamma(\xi \land i_\mu \gamma(\xi))$ and so $i_\mu \gamma$ is quasi-enlarging.

**Theorem 2.15.** Let $\mu \subset \mathcal{F}$ and $\gamma \in \Gamma$ be $\mu$-friendly. If $\nu \in \mu$ and $\xi$ is a $\gamma$-fuzzy open set, then $\xi \land \nu$ is again a $\gamma$-fuzzy open set.
Proof. Since $\xi$ is a $\gamma$–fuzzy open set, $\xi \subseteq \gamma(\xi)$. Then for $\nu \in \mu$, $\nu \land \xi \subseteq \nu \land (\gamma(\xi) \subseteq \gamma(\nu \land \xi))$ and so $\nu \land \xi$ is a $\gamma$–fuzzy open set.

Corollary 2.16. Let $\gamma \in \Gamma$, $\mu$ be the family of all $\gamma$–fuzzy open sets and $\gamma$ be $\mu$–friendly. Then $\lambda \land \nu \in \mu$ whenever $\lambda \in \mu$ and $\nu \in \mu$.

Corollary 2.16 leads to define a new subfamily of $\Gamma$, namely $\Gamma_4 = \{\gamma \in \Gamma \mid \gamma$ is $\mu_\gamma$–friendly\} where $\mu_\gamma$ is the family of all $\gamma$–fuzzy open sets. Hence, if $\gamma \in \Gamma_4$, then the GFTS $(X, \gamma)$ is closed under finite intersection, by Corollary 2.16. We call such spaces as Quasi-fuzzy topological spaces. Clearly, if $\gamma \in \Gamma_{14}$, then $\mu_\gamma$ is a fuzzy topological space. The following Example 2.17 shows that $\gamma \in \Gamma_4$ does not imply that $\gamma \in \Gamma_1$.

Example 2.17. Let $X = \mathbb{R}$, the set of all real numbers and $F$ be the family of all fuzzy sets defined on $X$. Define $\gamma : F \to F$ by $\gamma(\lambda) = \bar{\alpha}$ if $\bar{\alpha} \leq \lambda$, and $\gamma(\lambda) = 0$ if otherwise, where $0 < \alpha < 1$. Clearly, $\gamma \notin \Gamma_1$. Since $\{0, \bar{\alpha}\}$ is the family of all $\gamma$–fuzzy open sets, it follows that $\gamma \in \Gamma_4$.

Theorem 2.18. If $X$ is a nonempty set, $F$ is the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma_4$, then the following hold.
(a) $\iota, (\lambda \land \nu) = \iota, (\lambda \land \iota, (\nu)$ for every fuzzy sets $\lambda, \nu \in F$.

(b) $c_\gamma (\lambda \lor \nu) = c_\gamma (\lambda) \lor c_\gamma (\nu)$ for every fuzzy sets $\lambda, \nu \in F$.

Proof. (a) Since $i, (\gamma(\lambda) \leq \lambda$ and $i, (\gamma(\nu) \leq \nu$, by Corollary 2.16, $i, (\gamma(\lambda) \land i, (\gamma(\nu)$ is a $\gamma$–fuzzy open set contained in $\lambda \land \nu$ and so $i, (\gamma(\lambda) \land i, (\gamma(\nu) \leq i, (\gamma(\lambda \land \nu)$. Clearly, $i, (\gamma(\lambda \land \nu) \leq i, (\gamma(\lambda) \land i, (\gamma(\nu)$. This proves (a).

(b) Since $\lambda \lor \nu \leq c_\gamma (\lambda) \lor c_\gamma (\nu) \leq c_\gamma (\lambda \lor \nu)$, it follows that $c_\gamma (\lambda \lor \nu) = c_\gamma (\lambda) \lor c_\gamma (\nu)$ for every fuzzy sets $\lambda, \nu \in F$.

Lemma 2.19. Let $\lambda \in F$, $\gamma \in \Gamma$ and $\mu$ be the family of all $\gamma$–fuzzy open sets. Then a fuzzy point $x_t \in c_\gamma (\lambda)$ if and only if for every $\mu$–fuzzy open set $\nu$ of $x_t$, $\nu q_\lambda$.

Proof. Suppose $x_t \in c_\gamma (\lambda)$. Let $\nu$ be a $\mu$–fuzzy open set of $x_t$. If $\nu q_\lambda$, then $\lambda \leq (\bar{T} - \nu)$. Since $(\bar{T} - \nu)$ is $\mu$–fuzzy closed, $c_\gamma (\lambda) \leq (\bar{T} - \nu)$. Since $x_t \notin (\bar{T} - \nu)$, $x_t \notin c_\gamma (\lambda)$, a contradiction. Therefore, $\nu q_\lambda$. Conversely, suppose $x_t \notin c_\gamma (\lambda)$. Since $c_\gamma (\lambda) = \land \{\xi \mid \lambda \leq \xi \land \xi$ is $\mu$–fuzzy closed\}, there is a $\mu$–fuzzy closed set $\xi \supseteq \lambda$ such that $x_t \notin \xi$. Then $\bar{T} - \xi$ is a $\mu$–fuzzy open sets such that $x_t \in (\bar{T} - \xi)$. By hypothesis, $(\bar{T} - \xi)q_\lambda$. Since $\xi \geq \lambda, (\bar{T} - \xi)q_\lambda$, a contradiction to the hypothesis. Hence $x_t \in c_\gamma (\lambda)$.

Theorem 2.20. Let $\lambda \in F$, $\gamma \in \Gamma$ be $\lambda$–friendly and $\mu$ be the family of all $\gamma$–fuzzy open sets. Then $c_\gamma$ is $\lambda$–friendly.

Proof. Let $\nu \in \lambda, \xi \in F$ and $x_t \in \nu \land c_\mu (\xi)$. If $x_t \in \omega \in \mu$, then by Theorem 2.15, $\nu \land \omega$ is a $\gamma$–fuzzy open set containing $x_t$. By Lemma 2.19, $(\omega \land \nu)q_\xi$. Then clearly, $\omega q_\nu (\nu \land \xi)$ and so $x_t \in c_\mu (\nu \land \xi)$. Hence $\nu \land c_\mu (\xi) \leq c_\mu (\nu \land \xi)$ which implies that $c_\gamma$ is $\lambda$–friendly.

Corollary 2.21. If $X$ is a nonempty set, $F$ is the family of all fuzzy sets on $X$, $\gamma \in \Gamma_4$ and $\mu$ is the family of all $\gamma$–fuzzy open sets, then the following
hold.
(a) \( c_\gamma(\nu) \land \xi \leq c_\gamma(\nu \land \xi) \) for every fuzzy sets \( \nu, \xi \in \mu \).
(b) \( c_\gamma(c_\gamma(\nu) \land \xi) = c_\gamma(\nu \land \xi) \) for every fuzzy sets \( \nu, \xi \in \mu \).
(c) \( i_\gamma(\nu \lor \xi) \leq i_\gamma(\nu) \lor \xi \) for every fuzzy set \( \nu \) and \( \mu \)-fuzzy closed set \( \xi \).
(d) \( i_\gamma(\nu \lor \xi) = i_\gamma(i_\gamma(\nu) \lor \xi) \) for every fuzzy set \( \nu \) and \( \mu \)-fuzzy closed set \( \xi \).

**Proof.** (a) The proof follows from Theorem 2.20.
(b) Since \( \nu \land \xi \leq c_\gamma(\nu) \land \xi \), the proof follows from (a).
(c) If \( \xi \) is \( \mu \)-fuzzy closed, then \( \omega = \bar{1} - \xi \in \mu \) and so by (a), for \( \nu \in \mathcal{F} \), \( c_\gamma(\nu) \land \omega \leq c_\gamma(\nu \land \omega) \) and so \( \bar{1} - c_\gamma(\nu) \land \omega \leq \bar{1} - c_\gamma(\nu \land \omega) \). Therefore, \( i_\gamma(\bar{1} - (\nu \lor (1 - \omega))) \leq i_\gamma(\bar{1} - (\nu \lor (1 - \omega))) \leq i_\gamma(\bar{1} - \nu) \lor \xi \). If \( \psi = \bar{1} - \nu \), we have \( i_\gamma(\psi \lor \xi) \leq i_\gamma(\psi) \lor \xi \), which proves (c).
(d) The proof follows from (c).

**Corollary 2.22.** Let \( \lambda \subset \mathcal{F} \) be a GFT, \( \gamma \in \Gamma \) be \( \lambda \)-friendly and \( \mu \) be the family of all \( \gamma \)-fuzzy open sets. Then \( i_\mu c_\mu \) is quasi-enlarging.

**Proof.** The proof follows from Theorem 2.14 and Theorem 2.20.

In the rest of the section, we will consider a special type of enlargement whose domain is a subfamily of \( \mathcal{F} \). A function \( \kappa : \mu \rightarrow \mathcal{F} \) is an **enlargement** if \( \lambda \leq \kappa(\lambda) \) for every \( \lambda \in \mu \). The following are some examples of enlargements.

**Example 2.23.** Let \( X \) be a nonempty set, \( \mathcal{F} \) be the family of all fuzzy sets defined on \( X \) and \( \mu \subset \mathcal{F} \). Define \( \kappa : \mu \rightarrow \mathcal{F} \) by

(a) \( \kappa(\lambda) = \lambda \) for every \( \lambda \in \mu \).
(b) \( \kappa(\lambda) = c_\mu(\lambda) \) for every \( \lambda \in \mu \).
(c) \( \kappa(\lambda) = i_\mu c_\mu(\lambda) \) for every \( \lambda \in \mu \).

Then \( \kappa \) is an enlargement in each case.

Let \( \kappa : \mu \rightarrow \mathcal{F} \) is an enlargement. Define \( \kappa_\mu = \{ \lambda \in \mathcal{F} | \) For each \( x_t \in \lambda \), there exists \( \nu \in \mu \) such that \( x_t \in \nu \leq \kappa(\nu) \leq \lambda \} \). The following Theorem 2.24 gives some properties of \( \kappa_\mu \).

**Theorem 2.24.** Let \( X \) be a nonempty set, \( \mathcal{F} \) be the family of all fuzzy sets defined on \( X \), \( \mu \subset \mathcal{F} \) and \( \kappa : \mu \rightarrow \mathcal{F} \) be an enlargement. The following hold.

(a) \( \kappa_\mu \) is a GFT.
(b) If \( \mu \) is a GFT, then \( \kappa_\mu \subset \mu \).

**Proof.** (a) Clearly, \( \bar{0} \in \kappa_\mu \). Let \( \nu_\alpha \in \kappa_\mu \) for every \( \alpha \in \Delta \) and \( \nu = \lor\{ \nu_\alpha | \alpha \in \Delta \} \). If \( x_t \in \nu \), where \( t \in (0, 1] \), then \( x_t \in \nu_\alpha \) for some \( \alpha \in \Delta \). By hypothesis, there is a \( \xi \in \mu \) such that \( x_t \in \xi \leq \kappa(\xi) \leq \nu_\alpha \leq \nu \). Hence \( \nu \in \kappa_\mu \) which implies that \( \kappa_\mu \) is a GFT.
(b) Let \( \nu \in \kappa_\mu \). Then for each \( x_t \in \nu \) where \( t \in (0, 1] \), there exists \( \xi_x \in \mu \) such that \( \kappa(\xi_x) \leq \nu \) and so \( x_t \in \xi_x \leq \kappa(\xi_x) \leq \nu \). Hence \( \nu = \lor\{ \xi_x | x \in \nu \} \). Since \( \mu \) is a GFT, \( \nu \in \mu \) and so \( \kappa_\mu \subset \mu \).
References


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