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Operations in Generalized Fuzzy Topological Spaces

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Abstract

Generalized fuzzy topological space was introduced and studied by Palani Chetty in 2008. The aim of this paper is to define operations on the family of fuzzy sets of a set and discuss its properties.

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1. Introduction and Preliminaries

Let X be a nonempty set and $\mathcal{F} = \{\lambda \mid \lambda : X \to [0,1]\}$ be the family of all fuzzy sets defined on X. Let $\gamma : \mathcal{F} \to \mathcal{F}$ be a function such that $\lambda \leqslant \mu$ implies that $\gamma(\lambda) \leqslant \gamma(\mu)$ for every $\lambda,\ \mu \in \mathcal{F}$. That is, each γ is a monotonic function defined on $\mathcal{F}$. We will denote the collection of all monotonic functions defined on $\mathcal{F}$ by $\Gamma(\mathcal{F})$ or simply Γ. Let $\gamma \in \Gamma$. A fuzzy set $\lambda \in \mathcal{F}$ is said to be $\gamma-$*fuzzy open* [3] if $\lambda \leqslant \gamma(\lambda)$. Clearly, $\bar{0}$, the null fuzzy set is $\gamma-$fuzzy open. In [3], it is established that the arbitrary union of $\gamma-$fuzzy open sets is again a $\gamma-$fuzzy open set. A subfamily $\mathcal{G}$ of $\mathcal{F}$ is called a *generalized fuzzy topology*(GFT) [3] if $\bar{0} \in \mathcal{G}$ and $\vee\{\lambda_\alpha \mid \alpha \in \Delta\} \in \mathcal{G}$ whenever $\lambda_\alpha \in \mathcal{G}$ for every $\alpha \in \Delta$. If $\gamma \in \Gamma$, it follows that $\mathcal{A}$, the family of all $\gamma-$fuzzy open sets is a generalized

fuzzy topology. For $\lambda \in \mathcal{F}$, the $\gamma-$*interior* of λ, denoted by $i_\gamma(\lambda)$, is given by $i_\gamma(\lambda) = \vee\{\nu \in \mathcal{A} \mid \nu \leqslant \lambda\}$. Moreover, in [1], it is established that for all $\lambda \in \mathcal{F},\ i_\gamma(\lambda) \leqslant \lambda,\ i_\gamma i_\gamma(\lambda) = i_\gamma(\lambda)$ and $\lambda \in \mathcal{A}$ if and only if $\lambda = i_\gamma(\lambda)$. A fuzzy set $\lambda \in \mathcal{F}$ is said to be a $\gamma-$*fuzzy closed set* if $\bar{1} - \lambda$ is a $\gamma-$fuzzy open set. The intersection of all $\gamma-$fuzzy closed sets containing $\lambda \in \mathcal{F}$ is called the $\gamma-$*closure* of λ. It is denoted by $c_\gamma(\lambda)$ and is given by $c_\gamma(\lambda) = \wedge\{\mu \mid \bar{1} - \mu \in \mathcal{A},\ \lambda \leqslant \mu\}$. In [1], it is established that $c_\gamma(\lambda) = \bar{1} - i_\gamma(\bar{1} - \lambda)$ for all $\lambda \in \mathcal{F}$. A *fuzzy point* [4] x_α, with support $x \in X$ and value $0 < \alpha \leq 1$ is defined by $x_\alpha(y) = \alpha$, if $y = x$ and $x_\alpha(y) = 0$, if $y \neq x$. Again, for $\lambda \in \mathcal{F}$, we say that $x_\alpha \in \lambda$ if $\alpha \leqslant \lambda(x)$. Two fuzzy sets λ and β are said to be *quasi-coincident* [4], denoted by $\lambda q \beta$, if there exists $x \in X$ such that $\lambda(x) + \beta(x) > 1$ [4]. Two fuzzy sets λ and β are not quasi-coincident denoted by $\lambda \tilde{q} \beta$, if $\lambda(x) + \beta(x) \leq 1$ for all $x \in X$. Clearly, λ is a $\gamma-$fuzzy open set containing a point x_α if and only if $x_\alpha q \lambda$, and $\lambda \leqslant \beta$ if and only if $\lambda \tilde{q} (\overline{1} - \beta)$. For definitions not given here, refer [2].

2. Enlarging and quasi-Enlarging operations

Let X be a nonempty set and $\gamma \in \Gamma$. Let us agree in calling *operation*, any element of Γ. An operation $\gamma \in \Gamma$ is said to be *enlarging* if $\lambda \leqslant \gamma(\lambda)$ for every $\lambda \in \mathcal{F}$. If $\mathcal{B} \subset \mathcal{F}$, then $\gamma \in \Gamma$ is said to be $\mathcal{B}-$*enlarging* if $\lambda \leqslant \gamma(\lambda)$ for every $\lambda \in \mathcal{B}$. We will denote the family of all enlarging operations by Γ_e and the family of all $\mathcal{B}-$enlarging operations by $\Gamma_{\mathcal{B}}$. The easy proof of the following Theorem 2.1 is omitted.

Theorem 2.1. Let X be a nonempty set and $\mathcal{F}$ be the family of all fuzzy sets defined on X. If $\mathcal{C} \subset \mathcal{B} \subset \mathcal{F}$, then $\Gamma_{\mathcal{B}} \subset \Gamma_{\mathcal{C}}$. $\Gamma_e = \Gamma_{\mathcal{B}}$, if $\mathcal{B} = \mathcal{F}$.

An operation $\gamma \in \Gamma$, is said to be *quasi-enlarging* (QE) if $\gamma(\lambda) \leqslant \gamma(\lambda \wedge \gamma(\lambda))$ for every $\lambda \in \mathcal{F}$. An operation $\gamma \in \Gamma$, is said to be *weakly quasi-enlarging*(WQE) if $\lambda \wedge \gamma(\lambda) \leqslant \gamma(\lambda \wedge \gamma(\lambda))$ for every $\lambda \in \mathcal{F}$.If $\gamma \in \Gamma_e$, then $\lambda \wedge \gamma(\lambda) = \lambda$ for every $\lambda \in \mathcal{F}$ and so γ is quasi-enlarging. If γ is defined by $\gamma(\lambda) = \beta$ for every $\lambda \in \mathcal{F}$, then also γ is quasi-enlarging. If $\gamma \in \Gamma$ is quasi-enlarging, then it is weakly quasi-enlarging, since $\lambda \wedge \gamma(\lambda) \leqslant \gamma(\lambda) \leqslant \gamma(\lambda \wedge \gamma(\lambda))$. The following Example 2.2 shows that a weakly quasi-enlarging operation need not be a quasi-enlarging operation.

Example 2.2. Let $X = \{x, y, z\}$. Define $\gamma : \mathcal{F} \to \mathcal{F}$, by $\gamma(\lambda) = \bar{0}$, if $\lambda = \bar{0}$; $\gamma(\lambda) = \chi_{\{y\}}$, if $\lambda \leqslant \chi_{\{x\}}$; $\gamma(\lambda) = \chi_{\{z\}}$, if $\lambda \leqslant \chi_{\{z\}}$ and $\gamma(\lambda) = \bar{1}$ if otherwise. Then, $\lambda \wedge \gamma(\lambda) = \bar{0}$, if $\lambda = \bar{0}$; $\lambda \wedge \gamma(\lambda) = \bar{0}$, if $\lambda \leqslant \chi_{\{x\}}$; $\lambda \wedge \gamma(\lambda) \leqslant \chi_{\{z\}}$, if $\lambda \leqslant \chi_{\{z\}}$ and $\lambda \wedge \gamma(\lambda) = \lambda$ if otherwise. Therefore, $\gamma(\lambda \wedge \gamma(\lambda)) = \bar{0}$, if $\lambda = \bar{0}$; $\gamma(\lambda \wedge \gamma(\lambda)) = \bar{0}$, if $\lambda \leqslant \chi_{\{x\}}$; $\gamma(\lambda \wedge \gamma(\lambda)) = \chi_{\{z\}}$, if $\lambda \leqslant \chi_{\{z\}}$ and $\lambda \wedge \gamma(\lambda) = \bar{1}$, if otherwise and so it follows that γ is a weakly quasi-enlarging operator. If $\lambda = \chi_{\{x\}}$, then $\gamma(\lambda) = \chi_{\{y\}}$ but $\gamma(\lambda \wedge \gamma(\lambda)) = \gamma(\bar{0}) = \bar{0}$ and so γ is not a quasi-enlarging operator.

If $\gamma_1,\ \gamma_2 \in \Gamma$, then the composition $\gamma_1 \circ \gamma_2$ of the two operations γ_1 and γ_2 is again an operation and we write $\gamma_1\gamma_2$ instead of $\gamma_1 \circ \gamma_2$. The following Theorem 2.3 shows that the composition of enlarging operators is again an

enlarging operator and Theorem 2.5 below gives a property of quasi-enlarging operators.

Theorem 2.3. Let X be a nonempty set and $\mathcal{F}$ be the family of all fuzzy sets defined on X. If $\mathcal{B} \subset \mathcal{F}$, and γ_1 and γ_2 are $\mathcal{B}-$enlarging, then $\gamma_1\gamma_2$ is also $\mathcal{B}-$enlarging.

Proof. Suppose $\lambda \in \mathcal{B}$. Then $\lambda \leqslant \gamma_1(\lambda)$ and $\lambda \leqslant \gamma_2(\lambda)$. Now, $\lambda \leqslant \gamma_1(\gamma_2(\lambda))$, since $\gamma_1 \in \Gamma$. Therefore, $\gamma_1\gamma_2$ is $\mathcal{B}-$enlarging.

Corollary 2.4. If $\gamma_1,\ \gamma_2 \in \Gamma_e$, then $\gamma_1\gamma_2 \in \Gamma_e$.

Theorem 2.5. Let X be a nonempty set, $\mathcal{F}$ be the family of all fuzzy sets defined on X and $\mathcal{B} \subset \mathcal{F}$. If $\gamma \in \Gamma$ is quasi-enlarging, $\{\gamma(\lambda) \mid \lambda \in \mathcal{F}\} \subset \mathcal{B}$ and $\mu \in \Gamma_{\mathcal{B}}$, then $\mu\gamma$ is quasi-enlarging.

Proof. Let $\lambda \in \mathcal{F}$. Since γ is quasi-enlarging, $\gamma(\lambda) \leqslant \gamma(\lambda \wedge \gamma(\lambda))$. Since $\gamma(\lambda) \in \mathcal{B}$ and $\mu \in \Gamma_{\mathcal{B}},\ \gamma(\lambda) \leqslant \mu(\gamma(\lambda))$ and so $\gamma(\lambda) \leqslant \gamma(\lambda \wedge \mu\gamma(\lambda))$. Therefore, $\mu\gamma(\lambda) \leqslant \mu\gamma(\lambda \wedge \mu\gamma(\lambda))$. Hence $\mu\gamma$ is quasi-enlarging.

Theorem 2.6. Let X be a nonempty set and $\gamma \in \Gamma$. Then i_γ is quasi-enlarging and c_γ is enlarging.

Proof. If $\lambda \in \mathcal{F}$, then $i_\gamma(\lambda) = i_\gamma i_\gamma(\lambda) = i_\gamma(\lambda \wedge i_\gamma(\lambda))$, since $i_\gamma(\lambda) \leqslant \lambda$. So i_γ is quasi-enlarging. Again, $i_\gamma(\bar{1} - \lambda) \leqslant \bar{1} - \lambda$ and so $\lambda = \bar{1} - (\bar{1} - \lambda) \leqslant \bar{1} - i_\gamma(\bar{1} - \lambda) = c_\gamma(\lambda)$. Therefore, c_γ is enlarging.

Theorem 2.7. Let X be a nonempty set, $\gamma \in \Gamma$ and $\mathcal{A}$ be the family of all $\gamma-$fuzzy open sets. If $\mu \in \Gamma$, such that $i_\gamma\mu$ is quasi-enlarging and $\kappa \in \Gamma_{\mathcal{A}}$, then $\kappa i_\gamma \mu$ is quasi-enlarging.

Proof. If $\lambda \in \mathcal{F}$, then $i_\gamma\mu(\lambda) \in \mathcal{A}$. By Theorem 2.5, it follows that $\kappa i_\gamma\mu$ is quasi-enlarging.

Corollary 2.8. Let X be a nonempty set, $\gamma \in \Gamma$ and $\mathcal{A}$ be the family of all $\gamma-$fuzzy open sets. If $\kappa \in \Gamma_{\mathcal{A}}$, then κi_γ is quasi-enlarging.

Proof. If $\mu : \mathcal{F} \to \mathcal{F}$ is the identity operator, then $i_\gamma\mu = i_\gamma$ is quasi-enlarging and so the proof follows from Theorem 2.7.

Let $\{\gamma_\iota \in \Gamma \mid \iota \in \Delta \neq \emptyset\}$ be a family of operations. Define $\varphi : \mathcal{F} \to \mathcal{F}$ by $\varphi(\lambda) = \vee\{\gamma_\iota(\lambda) \mid \iota \in \Delta\}$ for every $\lambda \in \mathcal{F}$. The following Theorem 2.9 gives some properties of φ.

Theorem 2.9. Let X be a nonempty set. Let $\{\gamma_\iota \in \Gamma \mid \iota \in \Delta \neq \emptyset\}$ be a family of operations. Define $\varphi : \mathcal{F} \to \mathcal{F}$ by $\varphi(\lambda) = \vee\{\gamma_\iota(\lambda) \mid \iota \in \Delta\}$ for every $\lambda \in \mathcal{F}$. Then the following hold.

(a)$\varphi \in \Gamma$.

(b) If each γ_ι is $\mathcal{B}-$enlarging, then so is φ.

(c) If each γ_ι is quasi-enlarging, then so is φ.

(d) If each γ_ι is weakly quasi-enlarging, then so is φ.

Proof. (a) If $\lambda \leqslant \nu$, then $\gamma_\iota(\lambda) \leqslant \gamma_\iota(\nu)$ and so $\varphi(\lambda) = \vee\{\gamma_\iota(\lambda) \mid \iota \in \Delta\} \leqslant \vee\{\gamma_\iota(\nu) \mid \iota \in \Delta\} = \varphi(\nu)$. Therefore, $\varphi \in \Gamma$.

(b) Let $\lambda \in \mathcal{B}$. Then, by hypothesis, $\lambda \leqslant \gamma_\iota(\lambda)$ for every $\iota \in \Delta \neq \emptyset$. Therefore, $\lambda \leqslant \vee\gamma_\iota(\lambda) = \varphi(\lambda)$ and so φ is $\mathcal{B}-$enlarging.

(c) Suppose each γ_ι is quasi-enlarging. Then for $\lambda \in \mathcal{F}$, $\varphi(\lambda) = \vee\gamma_\iota(\lambda) \leqslant \vee\gamma_\iota(\lambda \wedge \gamma_\iota(\lambda)) \leqslant \vee\gamma_\iota(\lambda \wedge \varphi(\lambda)) = \varphi(\lambda \wedge \varphi(\lambda))$ and so φ is quasi-enlarging.
(d) For $\lambda \in \mathcal{F}$, $\lambda \wedge \varphi(\lambda) = \lambda \wedge (\vee\gamma_\iota(\lambda)) = \vee(\lambda \wedge \gamma_\iota(\lambda)) \leqslant \vee\gamma_\iota(\lambda \wedge \gamma_\iota(\lambda)) \leqslant \vee\gamma_\iota(\lambda \wedge \varphi(\lambda)) = \varphi(\lambda \wedge \varphi(\lambda))$. Therefore, φ is weakly quasi-enlarging.

Definition 2.10. Let X be a nonempty set and $\mathcal{A} \subset \mathcal{F}$. We say that an operation $\gamma \in \Gamma$ is $\mathcal{A}-$*friendly*, if $\nu \wedge \gamma(\lambda) \leqslant \gamma(\nu \wedge \lambda)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \mathcal{A}$.

The following Example 2.11 gives examples of $\mathcal{A}-$*friendly* operators. It is clear that if γ is $\mathcal{A}-$*friendly* and $\mathcal{B} \subset \mathcal{A}$, then γ is a $\mathcal{B}-$*friendly* operator. Theorem 2.12 below shows that the composition of friendly operators is again a friendly operator. Theorem 2.13 shows that arbitrary union of friendly operators is again a friendly operator.

Example 2.11. (a) If $\gamma : \mathcal{F} \to \mathcal{F}$ is defined by $\gamma(\lambda) = \theta$ for every $\lambda \in \mathcal{F}$ for some $\theta \in \mathcal{F}$, then γ is $\mathcal{A}-$friendly for every $\mathcal{A} \subset \mathcal{F}$.
(b) In any fuzzy topological space (X, τ), the fuzzy interior and closure operators i_τ and c_τ are $\tau-$friendly. That is, the following hold.
(i)$i_\tau(\lambda) \wedge \nu \leqslant i_\tau(\lambda \wedge \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \tau$.
(ii)$c_\tau(\lambda) \wedge \nu \leqslant c_\tau(\lambda \wedge \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \tau$.

Theorem 2.12. Let X be a nonempty set, γ, $\gamma_1 \in \Gamma$ and $\mathcal{A} \subset \mathcal{F}$. If γ and γ_1 are $\mathcal{A}-$friendly operators, then so is $\gamma_1\gamma$.

Proof. Suppose $\mathcal{A} \subset \mathcal{F}$ such that γ and γ_1 are $\mathcal{A}-$friendly. Then, $\gamma(\lambda) \wedge \nu \leqslant \gamma(\lambda \wedge \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \mathcal{A}$, and $\gamma_1(\lambda) \wedge \nu \leqslant \gamma_1(\lambda \wedge \nu)$ for every $\lambda \in \mathcal{F}$ and $\nu \in \mathcal{A}$. Replacing λ by $\gamma(\lambda)$ in the second inequality, we get $\gamma_1\gamma(\lambda) \wedge \nu \leqslant \gamma_1(\gamma(\lambda) \wedge \nu) \leqslant \gamma_1\gamma(\lambda \wedge \nu)$. Therefore, $\gamma_1\gamma$ is an $\mathcal{A}-$friendly operator.

Theorem 2.13. Let X be a nonempty set, $\mathcal{A} \subset \mathcal{F}$ and γ_ι is $\mathcal{A}-$friendly for every $\iota \in \Delta$. Then $\varphi = \vee\gamma_\iota$ is $\mathcal{A}-$friendly.

Proof. If $\lambda \in \mathcal{F}$, then for $\nu \in \mathcal{A}$, $\varphi(\lambda) \wedge \nu = (\vee\gamma_\iota)(\lambda) \wedge \nu = \vee(\gamma_\iota(\lambda)) \wedge \nu = \vee(\gamma_\iota(\lambda) \wedge \nu) \leqslant \vee\gamma_\iota(\lambda \wedge \nu) = \varphi(\lambda \wedge \nu)$. Therefore, φ is an $\mathcal{A}-$friendly operator.

Using friendly operators, next we construct quasi-enlarging operators using a generalized fuzzy topology (GFT). Let $\mu \subset \mathcal{F}$ be arbitrary. For $\lambda \in \mathcal{F}$, define $i_\mu(\lambda) = \vee\{\beta \in \mu \mid \beta \leqslant \lambda\}$ and $i_\mu(\lambda) = \bar{0}$, if no such $\beta \in \mu$ exists. Let $\mu' = \{\bar{1} - \lambda \mid \lambda \in \mu\}$. Define $c_\mu(\lambda) = \wedge\{\beta \in \mu \mid \lambda \leqslant \beta\}$ and $c_\mu(\lambda) = \bar{1}$, if no such $\beta \in \mu'$ exists. If μ is the family of all $\gamma-$open sets, then $c_\gamma = c_\mu$ and $i_\gamma = i_\mu$.

Theorem 2.14. Let $\mu \subset \mathcal{F}$ be a GFT. If $\gamma \in \Gamma$, is $\mu-$friendly, then $i_\mu\gamma$ is quasi-enlarging.

Proof. If $\xi \in \mathcal{F}$, then $i_\mu\gamma(\xi) = \gamma(\xi) \wedge i_\mu\gamma(\xi)$. Since $\gamma$is $\mu-$friendly, $\gamma(\xi) \wedge i_\mu\gamma(\xi) \leqslant \gamma(\xi \wedge i_\mu\gamma(\xi))$. Therefore, $i_\mu\gamma(\xi) = i_\mu i_\mu\gamma(\xi) \leqslant i_\mu\gamma(\xi \wedge i_\mu\gamma(\xi))$ and so $i_\mu\gamma$ is quasi-enlarging.

Theorem 2.15. Let $\mu \subset \mathcal{F}$ and $\gamma \in \Gamma$ be $\mu-$friendly. If $\nu \in \mu$ and ξ is a $\gamma-$fuzzy open set, then $\xi \wedge \nu$ is again a $\gamma-$fuzzy open set.

Proof. Since ξ is a $\gamma-$fuzzy open set, $\xi \leqslant \gamma(\xi)$. Then for $\nu \in \mu,\ \nu \wedge \xi \leqslant \nu \wedge \gamma(\xi) \leqslant \gamma(\nu \wedge \xi)$ and so $\nu \wedge \xi$ is a $\gamma-$fuzzy open set.

Corollary 2.16. Let $\gamma \in \Gamma,\ \mu$ be the family of all $\gamma-$fuzzy open sets and γ be $\mu-$friendly. Then $\lambda \wedge \nu \in \mu$ whenever $\lambda \in \mu$ and $\nu \in \mu$.

Corollary 2.16 leads to define a new subfamily of Γ, namely $\Gamma_4 = \{\gamma \in \Gamma \mid \gamma$ is $\mu_\gamma-$friendly$\}$ where μ_γ is the family of all $\gamma-$fuzzy open sets. Hence, if $\gamma \in \Gamma_4$, then the GFTS (X, γ) is closed under finite intersection, by Corollary 2.16. We call such spaces as *Quasi-fuzzy topological spaces*. Clearly, if $\gamma \in \Gamma_{14}$, then μ_γ is a fuzzy topological space. The following Example 2.17 shows that $\gamma \in \Gamma_4$ does not imply that $\gamma \in \Gamma_1$.

Example 2.17. Let $X = \mathbf{R}$, the set of all real numbers and $\mathcal{F}$ be the family of all fuzzy sets defined on X. Define $\gamma : \mathcal{F} \to \mathcal{F}$ by $\gamma(\lambda) = \bar{\alpha}$ if $\bar{\alpha} \leqslant \lambda, and\ \gamma(\lambda) = \bar{0}$ if otherwise, where $0 < \alpha < 1$. Clearly, $\gamma \notin \Gamma_1$. Since $\{\bar{0},\ \bar{\alpha}\}$ is the family of all $\gamma-$fuzzy open sets, it follows that $\gamma \in \Gamma_4$.

Theorem 2.18. If X is a nonempty set, $\mathcal{F}$ is the family of all fuzzy sets defined on X and $\gamma \in \Gamma_4$, then the following hold.
(a) $i_\gamma(\lambda \wedge \nu) = i_\gamma(\lambda) \wedge i_\gamma(\nu)$ for every fuzzy sets $\lambda,\ \nu \in \mathcal{F}$.
(b) $c_\gamma(\lambda \vee \nu) = c_\gamma(\lambda) \vee c_\gamma(\nu)$ for every fuzzy sets $\lambda,\ \nu \in \mathcal{F}$.

Proof. (a) Since $i_\gamma(\lambda) \leqslant \lambda$ and $i_\gamma(\nu) \leqslant \nu$, by Corollary 2.16, $i_\gamma(\lambda) \wedge i_\gamma(\nu)$ is a $\gamma-$fuzzy open set contained in $\lambda \wedge \nu$ and so $i_\gamma(\lambda) \wedge i_\gamma(\nu) \leqslant i_\gamma(\lambda \wedge \nu)$. Clearly, $i_\gamma(\lambda \wedge \nu) \leqslant i_\gamma(\lambda) \wedge i_\gamma(\nu)$. This proves (a).
(b) Since $\lambda \vee \nu \leqslant c_\gamma(\lambda) \vee c_\gamma(\nu) \leqslant c_\gamma(\lambda \vee \nu)$, it follows that $c_\gamma(\lambda \vee \nu) = c_\gamma(\lambda) \vee c_\gamma(\nu)$ for every fuzzy sets $\lambda,\ \nu \in \mathcal{F}$. □

Lemma 2.19. Let $\lambda \in \mathcal{F},\ \gamma \in \Gamma$ and μ be the family of all $\gamma-$fuzzy open sets. Then a fuzzy point $x_t \in c_\gamma(\lambda)$ if and only if for every $\mu-$fuzzy open set ν of $x_t,\ \nu q \lambda$.

Proof. Suppose $x_t \in c_\gamma(\lambda)$. Let ν be a $\mu-$fuzzy open set of x_t. If $\nu \tilde{q} \lambda$, then $\lambda \leqslant (\overline{1} - \nu)$. Since $(\overline{1} - \nu)$ is $\mu-$fuzzy closed, $c_\gamma(\lambda) \leqslant (\overline{1} - \nu)$. Since $x_t \notin (\overline{1} - \nu),\ x_t \notin c_\gamma(\lambda)$, a contradiction. Therefore, $\nu q \lambda$. Conversely, suppose $x_t \notin c_\gamma(\lambda)$. Since $c_\gamma(\lambda) = \wedge\{\xi \mid \lambda \leqslant \xi$ and ξ is $\mu-$fuzzy closed$\}$, there is a $\mu-$fuzzy closed set $\xi \geqslant \lambda$ such that $x_t \notin \xi$. Then $\overline{1} - \xi$ is a $\mu-$fuzzy open sets such that $x_t \in (\overline{1} - \xi)$. By hypothesis, $(\overline{1} - \xi) q \lambda$. Since $\xi \geqslant \lambda, (\overline{1} - \xi) \tilde{q} \lambda$, a contradiction to the hypothesis. Hence $x_t \in c_\gamma(\lambda)$.

Theorem 2.20. Let $\lambda \subset \mathcal{F},\ \gamma \in \Gamma$ be $\lambda-$friendly and μ be the family of all $\gamma-$fuzzy open sets. Then c_γ is $\lambda-$friendly.

Proof. Let $\nu \in \lambda, \xi \in \mathcal{F}$ and $x_t \in \nu \wedge c_\mu(\xi)$. If $x_t \in \omega \in \mu$, then by Theorem 2.15, $\nu \wedge \omega$ is a $\gamma-$fuzzy open set containing x_t. By Lemma 2.19, $(\omega \wedge \nu) q \xi$. Then clearly, $\omega q (\nu \wedge \xi)$ and so $x_t \in c_\mu(\nu \wedge \xi)$. Hence $\nu \wedge c_\mu(\xi) \leqslant c_\mu(\nu \wedge \xi)$ which implies that c_γ is $\lambda-$friendly.

Corollary 2.21 If X is a nonempty set, $\mathcal{F}$ is the family of all fuzzy sets on X, $\gamma \in \Gamma_4$ and μ is the family of all $\gamma-$fuzzy open sets, then the following

hold.
(a) $c_\gamma(\nu) \wedge \xi \leqslant c_\gamma(\nu \wedge \xi)$ for every fuzzy sets $\nu,\ \xi \in \mu$.
(b)$c_\gamma(c_\gamma(\nu) \wedge \xi) = c_\gamma(\nu \wedge \xi)$ for every fuzzy sets $\nu,\ \xi \in \mu$.
(c) $i_\gamma(\nu \vee \xi) \leqslant i_\gamma(\nu) \vee \xi$ for every fuzzy set ν and $\mu-$fuzzy closed set ξ.
(d) $i_\gamma(\nu \vee \xi) = i_\gamma(i_\gamma(\nu) \vee \xi)$ for every fuzzy set ν and $\mu-$fuzzy closed set ξ.

Proof. (a) The proof follows from Theorem 2.20.
(b). Since $\nu \wedge \xi \leqslant c_\gamma(\nu) \wedge \xi$, the proof follows from (a).
(c) If ξ is $\mu-$fuzzy closed, then $\omega = \bar{1} - \xi \in \mu$ and so by (a), for $\nu \in \mathcal{F},\ c_\gamma(\nu) \wedge \omega \leqslant c_\gamma(\nu \wedge \omega)$ and so $\bar{1} - c_\gamma(\nu \wedge \omega) \leqslant \bar{1} - (c_\gamma(\nu) \wedge \omega)$. Therefore, $i_\gamma((\bar{1} - \nu) \vee (\bar{1} - \omega)) \leqslant (\bar{1} - c_\gamma(\nu)) \vee (\bar{1} - \omega)$ and so $i_\gamma((\bar{1} - \nu) \vee \xi) \leqslant i_\gamma(\bar{1} - \nu) \vee \xi$. If $\psi = \bar{1} - \nu$, we have $i_\gamma(\psi \vee \xi) \leqslant i_\gamma(\psi) \vee \xi$, which proves (c).
(d) The proof follows from (c).

Corollary 2.22. Let $\lambda \subset \mathcal{F}$ be a GFT, $\gamma \in \Gamma$ be $\lambda-$friendly and μ be the family of all $\gamma-$fuzzy open sets. Then $i_\mu c_\mu$ is quasi-enlarging.

Proof. The proof follows from Theorem 2.14 and Theorem 2.20.

In the rest of the section, we will consider a special type of enlargement whose domain is a subfamily of $\mathcal{F}$. A function $\kappa : \mu \to \mathcal{F}$ is an *enlargement* if $\lambda \leqslant \kappa(\lambda)$ for every $\lambda \in \mu$. The following are some examples of enlargements.

Example 2.23. Let X be a nonempty set, $\mathcal{F}$ be the family of all fuzzy sets defined on X and $\mu \subset \mathcal{F}$. Define $\kappa : \mu \to \mathcal{F}$ by
(a)$\kappa(\lambda) = \lambda$ for every $\lambda \in \mu$.
(b) $\kappa(\lambda) = c_\mu(\lambda)$ for every $\lambda \in \mu$.
(c) $\kappa(\lambda) = i_\mu c_\mu(\lambda)$ for every $\lambda \in \mu$.
Then κ is an enlargement in each case.

Let $\kappa : \mu \to \mathcal{F}$ is an enlargement. Define $\kappa_\mu = \{\lambda \in \mathcal{F} \mid$ For each $x_t \in \lambda$, there exists $\nu \in \mu$ such that $x_t \in \nu \leqslant \kappa(\nu) \leqslant \lambda\}$. The following Theorem 2.24 gives some properties of κ_μ.

Theorem 2.24. Let X be a nonempty set, $\mathcal{F}$ be the family of all fuzzy sets defined on X, $\mu \subset \mathcal{F}$ and $\kappa : \mu \to \mathcal{F}$ be an enlargement. The the following hold.
(a)κ_μ is a GFT.
(b) If μ is a GFT, then $\kappa_\mu \subset \mu$.

Proof. (a) Clearly,$\overline{0} \in \kappa_\mu$. Let $\nu_\alpha \in \kappa_\mu$ for every $\alpha \in \triangle$ and $\nu = \vee\{\nu_\alpha \mid \alpha \in \triangle\}$. If $x_t \in \nu$, where $t \in (0, 1]$, then $x_t \in \nu_\alpha$ for some $\alpha \in \triangle$. By hypothesis, there is a $\xi \in \mu$ such that $x_t \in \xi \leqslant \kappa(\xi) \leqslant \nu_\alpha \leqslant \nu$. Hence $\nu \in \kappa_\mu$ which implies that κ_μ is a GFT.
(b) Let $\nu \in \kappa_\mu$. Then for each $x_t \in \nu$ where $t \in (0, 1]$, there exists $\xi_x \in \mu$ such that $\kappa(\xi_x) \leqslant \nu$ and so $x_t \in \xi_x \leqslant \kappa(\xi_x) \leqslant \nu$. Hence $\nu = \vee\{\xi_x \mid x \in \nu\}$. Since μ is a GFT, $\nu \in \mu$ and so $\kappa_\mu \subset \mu$.

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