On Lacunary Statistical Convergence of Double Sequences with Respect to the Intuitionistic Fuzzy Normed Space

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Abstract

In this paper, we study lacunary statistical convergence in intuitionistic fuzzy normed space. We also introduce here a new concept, that is, statistical completeness and show that IFNS is statistically complete but not complete.

Keywords: $t$-norm; $t$-conorm; intuitionistic fuzzy normed spaces; statistical convergence; lacunary statistical convergence; lacunary statistical Cauchy; statistical completeness

1 Introduction

The idea of statistical convergence was first introduced by Fast [3] and later on studied by many authors. The active researches on this topic were started after the paper of Fridy [5]. The concept of statistical convergence for fuzzy numbers has also been studied by various authors, e.g. [1], [2], [4], [14], [17].

Let $K$ be a subset of $\mathbb{N}$, the set of natural numbers. Then the asymptotic density of $K$ denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : k \in K \right\} \right|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be statistically convergent to the number $\ell$ if for each $\epsilon > 0$, the set $K(\epsilon) = \left\{ k \leq n : |x_k - \ell| > \epsilon \right\}$ has asymptotic density zero, i.e.

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - \ell| \geq \epsilon \right\} \right| = 0.$$

In this case we write $\text{st-lim} \ x = \ell$ [3], [5].
Note that every convergent sequence is statistically convergent to the same limit, but converse need not be true.

Statistical convergence of double sequences \( x = (x_{jk}) \) has been defined and studied by Mursaleen and Edely [10]; and for fuzzy numbers by Savaş and Mursaleen [14].

A real double sequence \( x = (x_{jk}) \) is said to be statistically convergent to the number \( \ell \) if for each \( \epsilon > 0 \), the set

\[
\{(j, k), j \leq m \text{ and } k \leq n : |x_{jk} - \ell| \geq \epsilon\}
\]

has double natural density zero. In this case we write \( st_2\lim x = \ell \) and we denote the set of all statistically convergent double sequences by \( S_2 \) and the set of all bounded statistically convergent double sequences by \( S_2^\infty \).

In the recent years, the theory of fuzzy has been on of the most active area of research in many branches of sciences. In 1965, this theory was introduced by Zadeh [18] and since then a huge number of research papers has published. In [12], Park introduced the concept of intuitionistic fuzzy metric space and later on Saadati and Park [12] introduced the concept of intuitionistic fuzzy normed space. Recently, Karakuş et al. [9] studied the concept of statistical convergence in intuitionistic fuzzy normed space, and Şençimen and Pehlivan [16] studied statistical convergence in fuzzy normed spaces. The concept of statistical convergence was introduced by Fast [3] which was later on studied by many authors. In [6], Fridy and Orhan introduced the idea of lacunary statistical convergence.

In this paper we shall study lacunary statistical convergence and lacunary statistical Cauchy for double sequences in intuitionistic fuzzy normed space.

2 Preliminaries

**Definition 2.1** [15] A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is said to be a continuous \( t \)-norm if it satisfies the following conditions:

1. \( * \) is associative and commutative,
2. \( * \) is continuous,
3. \( a * 1 = a \) for all \( a \in [0, 1] \),
4. \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) for each \( a, b, c, d \in [0, 1] \).
Definition 2.2 [15] A binary operation $\Diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous $t$-conorm if it satisfies the following conditions:

1. $\Diamond$ is associative and commutative,
2. $\Diamond$ is continuous,
3. $a \Diamond 0 = a$ for all $a \in [0, 1]$,
4. $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Using the continuous $t$-norm and $t$-conorm, Saadati and Park [13] have introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.3 The five-tuple $(X, \mu, \nu, *, \Diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if $X$ is a vector space, $*$ is a continuous $t$-norm, $\Diamond$ is a continuous $t$-conorm, and $\mu, \nu$ fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions: For every $x, y \in X$ and $s, t > 0$,

1. $\mu(x, t) + \nu(x, t) \leq 1$,
2. $\mu(x, t) > 0$,
3. $\mu(x, t) = 1$ if and only if $x = 0$,
4. $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
5. $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
6. $\mu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
7. $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$,
8. $\nu(x, t) < 1$,
9. $\nu(x, t) = 0$ if and only if $x = 0$,
10. $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
11. $\nu(x, t) \Diamond \nu(y, s) \geq \nu(x + y, t + s)$,
12. $\nu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
13. $\lim_{t \rightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x, t) = 1$.

In this case $(\mu, \nu)$ is called an intuitionistic fuzzy norm.
Example 2.1 Let \((X, \|\cdot\|)\) be a normed space, and let \(a \ast b = ab\) and \(a \diamond b = \min\{a + b, 1\}\) for all \(a, b \in [0, 1]\). For all \(x \in X\) and every \(t > 0\), consider
\[
\mu(x, t) := \frac{t}{t + \|x\|} \quad \text{and} \quad \nu(x, t) := \frac{\|x\|}{t + \|x\|}.
\]
Then \((X, \mu, \nu, \ast, \diamond)\) is an intuitionistic fuzzy normed space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed spaces are studied by Saadati and Park [13].

Definition 2.4 Let \((X, \mu, \nu, \ast, \diamond)\) be an IFNS. Then, a sequence \(x = (x_k)\) is said to be convergent to \(\ell \in X\) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for every \(\epsilon > 0\) and \(t > 0\), there exists \(k_0 \in \mathbb{N}\) such that \(\mu(x_k - \ell, t) > 1 - \epsilon\) and \(\nu(x_k - \ell, t) < \epsilon\) for all \(k \geq k_0\). In this case we write \((\mu, \nu)\)-\(\lim x = \ell\) or \(x_k \xrightarrow{\mu, \nu} \ell\) as \(k \to \infty\).

Definition 2.5. Let \((V, \mu, \nu, \ast, \diamond)\) be an IFNS. Then, \(x = (x_k)\) is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for every \(\epsilon > 0\) and \(t > 0\), there exists \(k_0 \in \mathbb{N}\) such that \(\mu(x_k - x_\ell, t) > 1 - \epsilon\) and \(\nu(x_k - x_\ell, t) < \epsilon\) for all \(k, \ell \geq k_0\).

Remark 2.1 [13]. Let \((X, \|\cdot\|)\) be a real normed linear space,
\[
\mu(x, t) := \frac{t}{t + \|x\|} \quad \text{and} \quad \nu(x, t) := \frac{\|x\|}{t + \|x\|}
\]
for all \(x \in X\) and \(t > 0\). Then \(x_n \xrightarrow{\|\cdot\|} x\) if and only if \(x_n \xrightarrow{\mu, \nu} x\).

3 Lacunary statistical convergence of double sequences in IFNS

In this section we study the concept of lacunary statistically convergent sequences in intuitionistic fuzzy normed space. First we define the concept of \(\theta\)-density:

Definition 3.1 By a lacunary sequence we mean an increasing integer sequence \(\theta = (K_r)\) such that \(K_0 = 0\) and \(h_r := k_r - k_{r-1} \to \infty\) as \(r \to \infty\). Throughout this paper the intervals determined by \(\theta\) will be denoted by \(I_r := \)
(k_{r-1}, k_r], and the ratio k_r/k_{r-1} will be abbreviated by q_r. Let K \subseteq \mathbb{N}. The number
\[ \delta_\theta(K) = \lim_{r \to \infty} \frac{1}{h_r} |\{k \in I_r : k \in K\}| \]
is said to be the \( \theta \)-density of \( K \), provided the limit exists.

**Definition 3.2** [6], [7] Let \( \theta \) be a lacunary sequence. Then a sequence \( x = (x_k) \) is said to be \( S_\theta \)-convergent to the number \( \ell \) if for every \( \epsilon > 0 \), the set \( K(\epsilon) \) has \( \theta \)-density zero, where
\[ K(\epsilon) := \{k \in I_r : |x_k - \ell| \geq \epsilon\}. \]
In this case we write \( S_\theta \)-lim \( x = \ell \) or \( x_k \to \ell(S_\theta) \).

Now we define the \( S_\theta \)-convergence of double sequences with respect to IFNS. For single sequences, see Mursaleen and Mohiuddine [11].

**Definition 3.3** Let \((X, \mu, \nu, *, \Diamond)\) be an IFNS and \( \theta \) be a lacunary sequence. Then, a sequence \( x = (x_{jk}) \) is said to be \( S_\theta \)-convergent to \( \ell \in X \) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if for every \( \epsilon > 0 \) and \( t > 0 \)
\[ \delta_\theta(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - \ell, t) \geq \epsilon\}) = 0, \quad (1) \]
or equivalently
\[ \delta_\theta(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) > 1 - \epsilon \text{ and } \nu(x_{jk} - \ell, t) < \epsilon\}) = 1. \quad (2) \]
In this case we write \( S_\theta^{(\mu, \nu)} \)-lim \( x = \ell \) or \( x_{jk} \to \ell(S_\theta) \), where \( \ell \) is said to be \( S_\theta^{(\mu, \nu)} \)-lim \( x \) and we denote the set of all \( S_\theta \)-convergent sequences with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) by \( S_\theta^{(\mu, \nu)} \).

Using (1) and (2), we easily get the following lemma.

**Lemma 3.1.** Let \((X, \mu, \nu, *, \Diamond)\) be an IFNS and \( \theta \) be a lacunary sequence. Then, for every \( \epsilon > 0 \) and \( t > 0 \), the following statements are equivalent:

1. \( S_\theta^{(\mu, \nu)} \)-lim \( x = \ell \).
2. \( \delta_\theta(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) \leq 1 - \epsilon\}) = \delta_\theta(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - \ell, t) \geq \epsilon\}) = 0. \)
3. \( \delta_\theta(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) > 1 - \epsilon \text{ and } \nu(x_{jk} - \ell, t) < \epsilon\}) = 1. \)
4. $\delta_{\theta}\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) > 1 - \epsilon\} = \delta_{\theta}\{(k \in \nu(x_{jk} - \ell, t) < \epsilon)\} = 1$.

5. $S_{\theta}\lim \mu(x_{jk} - \ell, t) = 1$ and $S_{\theta}\lim \nu(x_{jk} - \ell, t) = 0$.

**Theorem 3.2.** Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS and $\theta$ be a lacunary sequence. If a sequence $x = (x_{jk})$ is lacunary statistically convergent with respect to the intuitionistic fuzzy norms $(\mu, \nu)$, then $S^{(\mu, \nu)}_{\theta}$-limit is unique.

**Proof.** Suppose that $S^{(\mu, \nu)}_{\theta}\lim x = \ell_1$ and $S^{(\mu, \nu)}_{\theta}\lim x = \ell_2$. Given $\epsilon > 0$ choose $s > 0$ such that $(1 - s) * (1 - s) > 1 - \epsilon$ and $s \Diamond s < \epsilon$. Then, for any $t > 0$, define the following sets as:

\[ K_{\mu,1}(s, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell_1, t) \leq 1 - s\}, \]
\[ K_{\mu,2}(s, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell_2, t) \leq 1 - s\}, \]
\[ K_{\nu,1}(s, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell_1, t) \geq s\}, \]
\[ K_{\nu,2}(s, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell_2, t) \geq s\}. \]

Since $S^{(\mu, \nu)}_{\theta}\lim x = \ell_1$, we have using Lemma 3.1

\[ \delta_{\theta}(K_{\mu,1}(\epsilon, t)) = \delta_{\theta}(K_{\nu,1}(\epsilon, t)) = 0 \text{ for all } t > 0. \]

Furthermore, using $S^{(\mu, \nu)}_{\theta}\lim x = \ell_2$, we get

\[ \delta_{\theta}(K_{\mu,2}(\epsilon, t)) = \delta_{\theta}(K_{\nu,2}(\epsilon, t)) = 0 \text{ for all } t > 0. \]

Now let $K_{\mu,\nu}(\epsilon, t) = (K_{\mu,1}(\epsilon, t) \cup K_{\mu,2}(\epsilon, t)) \cap (K_{\nu,1}(\epsilon, t) \cup K_{\nu,2}(\epsilon, t))$. Then observe that $\delta_{\theta}(K_{\mu,\nu}(\epsilon, t)) = 0$ which implies $\delta_{\theta}(\mathbb{N} \setminus K_{\mu,\nu}(\epsilon, t)) = 1$. If $k \in \mathbb{N} \setminus K_{\mu,\nu}(\epsilon, t)$, then we have two possible cases. (a) $k \in \mathbb{N} \setminus (K_{\mu,1}(\epsilon, t) \cup K_{\mu,2}(\epsilon, t))$, and (b) $k \in \mathbb{N} \setminus (K_{\nu,1}(\epsilon, t) \cup K_{\nu,2}(\epsilon, t))$. We first consider that $k \in \mathbb{N} \setminus (K_{\mu,1}(\epsilon, t) \cup K_{\mu,2}(\epsilon, t))$. Then we have

\[ \mu(\ell_1 - \ell_2, t) \geq \mu(x_k - \ell_1, \frac{t}{2}) \ast \mu(x_k - \ell_2, \frac{t}{2}) > (1 - s) * (1 - s). \]

Since $(1 - s) * (1 - s) > 1 - \epsilon$, it follows that

\[ \mu(\ell_1 - \ell_2, t) > 1 - \epsilon. \]

Since $\epsilon > 0$ was arbitrary, we get $\mu(\ell_1 - \ell_2, t) = 1$ for all $t > 0$, which yields $\ell_1 = \ell_2$. On the other hand, if $k \in \mathbb{N} \setminus (K_{\mu,1}(\epsilon, t) \cup K_{\nu,2}(\epsilon, t))$, then we may write

\[ \nu(\ell_1 - \ell_2, t) \leq \nu(x_k - \ell_1, \frac{t}{2}) \Diamond \nu(x_k - \ell_2, \frac{t}{2}) < s \Diamond s. \]
Now using the fact that \( s \diamond s < \epsilon \), we see that

\[
\nu(\ell_1 - \ell_2, t) < \epsilon.
\]

So we have \( \nu(\ell_1 - \ell_2, t) = 0 \) for all \( t > 0 \), which implies \( \ell_1 = \ell_2 \). Therefore, in all cases, we conclude that \( S_{\theta}^{(\mu, \nu)} \)-limit is unique.

This completes the proof of the theorem.

**Theorem 3.3.** Let \((X, \mu, \nu, *, \diamond)\) be an IFNS and \( \theta \) be any lacunary sequence. If \( (\mu, \nu)\)-\( \lim x = \ell \) then \( S_{\theta}^{(\mu, \nu)} \)-\( \lim x = \ell \).

**Proof.** Let \( (\mu, \nu)\)-\( \lim x = \ell \). Then for every \( \epsilon > 0 \) and \( t > 0 \), there is a number \( k_0 \in \mathbb{N} \) such that

\[
\mu(x_k - \ell, t) > 1 - \epsilon \quad \text{and} \quad \nu(x_k - \ell, t) < \epsilon
\]

for all \( k \geq k_0 \). Hence the set

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) \leq 1 - \epsilon \quad \text{or} \quad \nu(x_{jk} - \ell, t) \geq \epsilon\}
\]

has finite number of terms. Since every finite subset of \( \mathbb{N} \) has density zero and hence

\[
\delta_{\theta}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, t) \leq 1 - \epsilon \quad \text{or} \quad \nu(x_{jk} - \ell, t) \geq \epsilon\}) = 0,
\]

that is, \( S_{\theta}^{(\mu, \nu)} \)-\( \lim x = \ell \). This completes the proof of the theorem.

### 4 Lacunary statistically Cauchy double sequences in IFNS

In [9], Karakus has defined the concept of statistically Cauchy sequences on intuitionistic fuzzy normed space. In this section we define lacunary statistically Cauchy double sequences with respect to an intuitionistic fuzzy normed space and introduce a new concept of statistical completeness.

**Definition 4.1.** Let \((X, \mu, \nu, *, \diamond)\) be an IFNS and \( \theta \) be a lacunary sequence. Then, a sequence \( x = (x_{jk}) \) is said to be *lacunary statistically Cauchy* (or \( S_{\theta}\)-Cauchy) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if for every \( \epsilon > 0 \) and \( t > 0 \), there exist \( N = N(\epsilon) \) and \( M = M(\epsilon) \) such that

\[
\delta_{\theta}(\{k \in \mathbb{N} : \mu(x_{jk} - x_{MN}, t) \leq 1 - \epsilon \quad \text{or} \quad \nu(x_{jk} - x_{MN}, t) \geq \epsilon\}) = 0.
\]
Theorem 4.1. Let \((X, \mu, \nu, *, \Diamond)\) be an IFNS and \(\theta\) be any lacunary sequence. A sequence \(x = (x_{jk})\) is \(S_\theta\)-convergent if and only if it is \(S_\theta\)-Cauchy with respect to the intuitionistic fuzzy norm \((\mu, \nu)\).

Proof. Let \(x = (x_k)\) be \(S_\theta\)-convergent to \(\ell\) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\), i.e., \(S_\theta(\mu, \nu)\)-lim \(x = \ell\). Then

\[
\delta_\theta\left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, \frac{t}{2}) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - \ell, \frac{t}{2}) \geq \epsilon\}\right) = 0.
\]

In particular, for \(k = N\)

\[
\delta_\theta\left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{MN} - \ell, \frac{t}{2}) \leq 1 - \epsilon \text{ or } \nu(x_{MN} - \ell, \frac{t}{2}) \geq \epsilon\}\right) = 0.
\]

Since

\[
\mu(x_{jk} - x_{MN}, t) = \mu(x_{jk} - \ell - x_{MN} + \ell, \frac{t}{2} + \frac{t}{2}) \\
\geq \mu(x_{jk} - \ell, \frac{t}{2}) * \mu(x_{MN} - \ell, \frac{t}{2})
\]

and since

\[
\nu(x_{jk} - x_{MN}, t) \leq \nu(x_{jk} - \ell, \frac{t}{2}) \Diamond \nu(x_{MN} - \ell, \frac{t}{2}),
\]

we have

\[
\delta_\theta\left(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{MN}, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - x_{MN}, t) \geq \epsilon\}\right) = 0,
\]

that is, \(x\) is \(S_\theta\)-Cauchy with respect to the intuitionistic fuzzy norm \((\mu, \nu)\).

Conversely, let \(x = (x_{jk})\) be \(S_\theta\)-Cauchy but not \(S_\theta\)-convergent with respect to the intuitionistic fuzzy norm \((\mu, \nu)\). Then there exists \(N\) such that

\[
\delta_\theta(A(\epsilon, t)) = 0, \quad (3)
\]

\[
\delta_\theta(B(\epsilon, t)) = 0, \quad \text{i.e.} \quad \delta_\theta(B^C(\epsilon, t)) = 1; \quad (4)
\]

where

\[
A(\epsilon, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{MN}, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - x_{MN}, t) \geq \epsilon\},
\]

\[
B(\epsilon, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, \frac{t}{2}) > \frac{1 - \epsilon}{2} \text{ and } \nu(x_{jk} - \ell, \frac{t}{2}) < \frac{\epsilon}{2}\}.
\]

Since

\[
\mu(x_{jk} - x_{MN}, t) \geq 2\mu(x_{jk} - \ell, \frac{t}{2}) > 1 - \epsilon,
\]

\[
\nu(x_{jk} - x_{MN}, t) \leq \nu(x_{jk} - \ell, \frac{t}{2}) \Diamond \nu(x_{MN} - \ell, \frac{t}{2}),
\]

\[
\delta_\theta(B(\epsilon, t)) = 0.
\]

Therefore, \(x\) is \(S_\theta\)-Cauchy with respect to the intuitionistic fuzzy norm \((\mu, \nu)\).
and
\[ \nu(x_{jk} - x_{MN}, t) \leq 2\nu(x_{jk} - \ell, \frac{t}{2}) < \epsilon, \]
if \( \mu(x_{jk} - \ell, \frac{t}{2}) > (1 - \epsilon)/2 \) and \( \nu(x_{jk} - \ell, \frac{t}{2}) < \epsilon/2 \). Therefore
\[ \delta_{\theta}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{MN}, t) > 1 - \epsilon \text{ and } \nu(x_{jk} - x_{MN}, t) < \epsilon\}) = 0, \]
that is, \( \delta_{\theta}(A(\epsilon, t)) = 1 \), which contradicts (3), since \( x \) was \( S_\theta \)-Cauchy with respect to intuitionistic fuzzy norm \( (\mu, \nu) \). So that \( x \) must be \( S_\theta \)-convergent with respect to intuitionistic fuzzy norm \( (\mu, \nu) \).

References


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