On QFGP-Injective Modules

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Abstract

In this paper, we give some characterizations and properties of QFGP-injective modules. It is shown that if $M_R$ is a QFGP-injective $R$-module with $S = \text{End}(M_R)$, then for any $0 \neq a \in S$, there exists $0 \neq c \in S$, such that $ac = ca \neq 0$ and $l_S(c(M) \cap \text{Ker}(a)) = Sa + l_S(c(M))$; if for any $s \in S$, $s(M)$ is projective, and there exist $0 \neq t \in S$ such that $0 \neq st = ts$, and $l_S(\text{Ker}(st)) = Sst$, then $st$ is a regular element of $S$; if $M_R$ is a QFGP-injective module and $S$ its endomorphism ring, then for any right uniform element of $S$, the set $A_u = \{s \in S|Kersu(M) \neq 0\}$ is a maximal left ideal of $S$ containing $l_S(u(M))$; if $M_R$ is a QFGP-injective module and $W = \oplus_{i=1}^{n}u_i(M)$ a direct sum of uniform submodule $u_i(M)$ of $M$, and $A \subset S$ is a maximal left ideal which is not of the form $A_u$ for some right uniform element $u$ of $S$, then there is $\psi \in A$ such that $\text{Ker}(1 - \psi) \cap W$ is essential in $W$.

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1 Introduction.

Throughout the paper, $R$ will be an associative ring with identity and $M$ is a right $R$-module with $S = \text{End}(M_R)$. For a subset $X$ of $R$, the left(right) annihilator of $X$ in $R$ is denote by $l(X)(r(X))$. If $X = \{a\}$, we usually

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abbreviate it to \(l(a)(r(a))\). For a set \(A\) of \(M\), the left annihilator of \(A\) in \(S\) is denote by \(l_S(A)\). As usual, we denote the socle and the Jacobson radical of of a module \(N\) by \(\text{Soc}(N)\) and \(\text{Rad}N\), respectively. We write \(J(R)\) and \(Z(R_R)/(Z(R_R))\) for the Jacobson radical of \(R\) and right(left) singular ideal of \(R\) respectively.

Recall a ring \(R\) is called right principally injective \([1]\)(or P-injective for short), if every homomorphism from a principally right ideal of \(R\) to \(R\) can be extended to an endomorphism of \(R\), or equivalently, \(lr(a) = Ra\) for all \(a \in R\). The notion of right P-injective rings has been generalized by many authors. For example, in \([2]\), right P-injective rings are generalized to quasi P-injective modules. A right \(R\)-module \(M\) is called quasi P-injective, if each \(R\)-homomorphism from an \(M\)-cyclic submodule \(s(M)\) of \(M\) to \(M\) can be extended to an endomorphism of \(M\). This is equivalent to saying that if \(l_S(\text{Kers}) = Ss\) for all \(s \in S\). In \([3]\), right P-injective rings are generalized to almost principally injective rings, that is, a ring \(R\) is said to be almost principally injective (or AP-injective for short), if for any \(a \in R\), there exists a left ideal \(X_a\) such that \(lr(a) = Ra \oplus X_a\). In \([4]\), the authors defined almost quasi P-injective modules in the above similar ways to AP-injective rings. In \([5]\), FGP-injective rings are studied. A ring \(R\) is called right FGP-injective, if for any \(0 \neq a \in R\), there exists \(0 \neq c \in R\) such that \(0 \neq ac = ca\) and any right \(R\)-homomorphism \(f : acR \rightarrow R\) can be extended to \(R \rightarrow R\). The nice structure of FGP-injective rings draws our attention to define quasi-FGP-injective modules, and to investigate their characterizations and properties.

2 Main Results.

**Definition 2.1** Let \(M\) be a right \(R\)-module, \(S = \text{End}(M_R)\), the module \(M\) is called right quasi FGP-injective (or QFGP-injective for short), if for any \(0 \neq a \in S\), there exists \(0 \neq c \in S\) such that \(0 \neq ac = ca\) and any right \(R\)-homomorphism from \(ac(M)\) to \(M\) extends to an endomorphism of \(M\).

**Theorem 2.2** Let \(M_R\) be a right \(R\)-module with \(S = \text{End}(M_R)\). Then the following stations are equivalent.

1. \(M\) is right FGP-injective.
2. For any \(0 \neq a \in S\), there exists \(0 \neq c \in S\), such that \(ac = ca \neq 0\), and \(l_S(\text{Ker}(ac)) = \text{Soc}\).

**Proof** (1) ⇒ (2). By (1), for any \(0 \neq a \in S\), there exists \(0 \neq c \in S\), such that \(0 \neq ac = ca\), and for any \(R\)-homomorphism \(ac(M) \rightarrow M\) can be extended to an endomorphism of \(M\). For any \(t \in l_S(\text{Ker}(ac)), t(\text{Ker}(ac)) = 0, \text{Ker}(ac) \subseteq \text{Kert}\). Let \(s_1 : M \rightarrow ac(M), t_1 : M \rightarrow t(M)\) be \(R\)-homomorphism induced by \(ac\) and \(t\), respectively, and \(i_1 : ac(M) \rightarrow M\) and \(i_2 : t(M) \rightarrow M\) the embeddings. Since \(s_1\) is an epimorphism, there is an \(R\)-homomorphism \(\varphi :\)
then for any \( 0 \neq x \in \mathbb{F} \), there exists \( 0 \neq c \in S \), such that \( ac = ca \neq 0 \), and \( l_s(Ker(ac)) = Sac \). For any \( R \)-homomorphism \( \varphi : ac(M) \rightarrow M \) such that \( wt_1 = i_2 \varphi \). Hence \( t = uac \) and \( t \in Sac \). On the other hand, since \( ac \in l_s(Ker(ac)) \), \( Sac \subseteq l_s(Ker(ac)) \).

(2)⇒(1). By (2), for any \( 0 \neq a \in S \), there exists \( 0 \neq c \in S \), such that \( ac = ca \neq 0 \), and \( l_s(Ker(ac)) = Sac \). For any \( R \)-homomorphism \( \varphi : ac(M) \rightarrow M \), let \( \pi : M \rightarrow ac(M) \) be a natural epimorphism, then \( \varphi \pi \) is an \( R \)-homomorphism from \( M \) to \( M \), and \( Ker(ac) \subseteq Ker(\varphi \pi) \), so \( \varphi \pi \in l_s(Ker(\varphi \pi)) \subseteq l_s(Ker(ac)) = Sac \). Hence there exists \( u \in S \) such that \( \varphi \pi = uac \).

**Corollary 2.3** Let \( M_R \) be a QFGP-injective \( R \)-module with \( S = End(M_R) \), then for any \( 0 \neq a \in S \), there exists \( 0 \neq c \in S \), such that \( ac = ca \neq 0 \) and \( l_s(c(M) \cap Ker(a)) = Sa + l_s(c(M)) \).

**Proof** For any \( 0 \neq a \in S \), by theorem 2.2, there exists \( 0 \neq c \in S \) such that \( ac = ca \neq 0 \) and \( l_s(Ker(ac)) = Sac \). Since \( l_s(c(M)) = 0 \), \( l_s(c(M)) \subseteq l_s(c(M) \cap Kera) \). And \( Sa(Kera) = 0 \), so \( Sa \subseteq l_s(Kera) \subseteq l_s(Kera \cap c(M)) \). Hence \( Sa + l_s(c) \subseteq l_s(Kera \cap c(M)) \). On the other hand, for any \( x \in l_s(Kera \cap c(M)) \), it is easy to see that \( Ker(ac) \subseteq Ker(xc) \), so \( xc \in l_s(Kera(xc)) \subseteq l_s(Kera(ac)) = Sac \), thus \( xc = yac, y \in S \), then \( (x - ya)c = 0 \), \( (x - ya) \in l_s(c(M)) \). Hence \( x = ya + (x - ya) \in Sa + l_s(c(M)) \), that is \( l_s(Kera \cap c(M)) \subseteq Sa + l_s(c(M)) \). The conclusion is proved.

**Theorem 2.4** Let \( M_R \) be a right \( R \)-module with \( S = End(M_R) \). Then
(1) If \( S \) is right FGP-injective, then \( M_R \) is QFGP-injective.
(2) If \( M_R \) is QFGP-injective, and \( M \) generates \( Ker \) for all \( s \in S \), then \( S \) is right FGP-injective.

**Proof** (1) Let \( 0 \neq s \in S \). Since \( S \) is right FGP-injective, there exists \( 0 \neq t \in S \) such that \( st = ts \neq 0 \) and \( l_s r_s(st) = Sst \). If \( a \in l_s(Ker(st)) \), and \( b \in r_s(st) \), then \( stb = 0 \), so \( b(M) \subseteq Ker(st) \), and hence \( ab(M) = 0 \), that is \( ab = 0 \). It follows that \( l_s(Ker(st)) \subseteq l_s r_s(st) \). Thus, we have \( Sst \subseteq l_s(Ker(st)) \subseteq Sst \). So \( l_s(Ker(st)) = Sst \), and (1) is proved.

(2) Let \( 0 \neq s \in S \). Since \( M_R \) is QFGP-injective, there exists \( 0 \neq t \in S \) such that \( st = ts \neq 0 \) and \( l_s(Ker(st)) = Sst \). Assume \( a \in l_s r_s(st) \). Since \( M \) generates \( Ker(st) \), \( Ker(st) = \sum_{u \in T} u(M) \) for some subset \( T \) of \( S \). It is easy to see that \( au = 0 \) for each \( u \in T \), thus \( ax = 0 \) for each \( x \in Ker(st) \). This implies that \( l_s r_s(st) \subseteq l_s(Ker(st)) \). Hence \( Sst \subseteq l_s r_s(st) \subseteq l_s(Ker(st)) = Sst \), thus \( Sst = l_s r_s(st) \). Therefore \( S \) is right FGP-injective.

**Theorem 2.5** Let \( M_R \) be a QFGP-injective module, for any \( x \in S \), if \( x(M) \) is a minimal right submodule of \( M \), then \( Sx \) is a minimal left ideal of \( S \).

**Proof** Suppose that \( x(M) \) is a minimal right submodule. Let \( 0 \neq y = ax \in Sx \). Since \( y \neq 0 \), there exists \( 0 \neq c \) such that \( 0 \neq yc = cy \), and any homomorphism \( yc(M) \rightarrow M \) can be extended to an endomorphism of \( M \). Define \( h : x(M) \rightarrow yc(M) \) such that \( h(x(m)) = yc(m) \), for \( m \in M \). This is a well-defined homomorphism. Note that \( Kerh \neq x(M) \), and since \( x(M) \) is
minimal, then $\text{Ker} h = 0$, so $h$ is an isomorphism. Let $i : x(M) \to M$ be the inclusion map. Then $f = ih^{-1}$ is a homomorphism from $yc(M) \to M$, and $f(yc(m)) = x(m), m \in M$. Since $f$ can be extended to an endomorphism of $M$, then there exists $s \in S$ such that $x = syc = dy, d = sc$. Therefore $Sx = Sy$, which shows that $Sx$ is a minimal left ideal of $S$.

**Lemma 2.6** Let $M_R$ be a right QGP-injective module, then for any minimal left ideal $S_1$ of $S$, $l_S(r_M(S_1)) = S_1$.

**Proof** Since $S_1$ is a minimal left ideal, then for any $0 \neq a \in S_1$, $S_1 = Sa$. By hypothesis, there exists $0 \neq c \in S$, such that $ac = ca \neq 0$ and $l_S(r_M(ac)) = Sac$. Since $Sa$ is minimal, $Sac = Sa$. Thus $S_1 = Sa = Sac = l_S(r_M(ac)) = l_S(r_M(Sac)) = l_S(r_M(S_1))$.

**Corollary 2.7** Let $R$ be a right FGP-injective ring, if $K$ is a minimal left ideal of $R$, then $lr(K) = K$.

**Lemma 2.8** (see [6, Lemma 11 and Theorem 12]) Let $M_R$ be a Kasch module with $S = \text{End}(M_R)$, then $r_Ml_S(T) = T$ for all maximal submodules $T$ of $M$.

**Theorem 2.9** Let $M_R$ be a QGP-injective module, and $M_R$ is finitely generated Kasch module, then the mappings

$$K \to r_M(K) \text{ and } T \to l_S(T)$$

are mutually inverse bijections between the set of all minimal left ideals $K$ of $S$ and the set of all maximal submodules $T$ of $M$.

**Proof** Claim 1. $r_M(K)$ is maximal for all minimal left ideals $K$ of $S$. In fact, let $r_M(K) \subseteq T$, where $T$ is a maximal submodule of $M$, then $l_S(T) \subseteq l_S(r_M(K)) = K$ by Lemma 2.6, so $l_S(T) = K$ since $M$ is Kasch, hence $r_M(K) = r_Ml_S(T) = T$ by Lemma 2.8.

Claim 2. $l_S(T)$ is minimal for all maximal submodules $T$ of $M_R$. In fact, $l_S(T) \neq 0$ since $M$ is Kasch. For any $0 \neq s \in l_S(T)$, there exists $0 \neq t \in S$ such that $0 \neq st = ts$ and $l_S(r_M(st)) = Sst$. Then $T = r_Ml_S(T) = r_M(st)$, hence $T = r_M(st)$. Thus $l_S(T) = l_S(r_M(st)) = Sst$. Observe that $Sst \subseteq Ss \subseteq l_S(T)$, so $l_S(T) = Ss$. It follows that $l_S(T)$ is a minimal left ideal of $S$.

**Theorem 2.10** Let $M_R$ be a right $R$-module, suppose for any $s \in S$, $s(M)$ is projective, if there exist $0 \neq t \in S$ such that $0 \neq st = ts$, and $l_S(\text{Ker}(st)) = Sst$, then $st$ is a regular element of $S$.

**Proof** For any $s \in S$, there exists $0 \neq t \in S$ such that $st = ts \neq 0$ and $l_S(\text{Ker}(st)) = Sst$. Let $\varphi : M \to st(M); \varphi(m) = st(m)$, since $st(M)$ is projective, then there exists a short exact sequence: $0 \to \text{Ker}(st) \to M \to st(M) \to 0$. Thus $\text{Ker}(st)$ is a direct summand of $M$. Write $\text{Ker}(st) = e(M), e = e^2$. So $l_S(\text{Ker}(st)) = S(1 - e)$, let $1 - e = f$, then $f = f^2$. Since $st \in l_S(\text{Ker}(st)) = Sf$, thus $st = yf, y \in S, stf = st$. By hypothesis, $l_S(\text{Ker}(st)) = Sst$, so $f = zst, z \in S$, thus $st = stf = stzst$, hence $st$ is a regular element of $S$.

**Corollary 2.11** Let $R$ be a right PP ring, for any $0 \neq a \in R$, if there
exists \( 0 \neq b \in R \) such that \( 0 \neq ab = ba \), and \( lr(ab) = Rab \), then \( ab \) is a regular element of \( R \).

**Theorem 2.12** Let \( M_R \) be a QFGP-injective module which is a self-generator, then \( J(S) = \Delta \), where \( \Delta = \{ s \in S | \text{Ker}s \text{ is essential in } M \} \).

**Proof** Let \( x \in J(S) \), then we will show \( x \in \Delta \). If not, then there exists \( 0 \neq m \in M \) such that \( \text{Ker} x \cap mR = 0 \). Since \( M \) is a self-generator, there exists a subset \( T \subseteq S \) such that \( mR = \sum_{t \in T} t(M) \), and \( mR \neq 0 \) implies that there exists \( s \in T \) such that \( xs \neq 0 \), then by QFGP-injectivity, there exist \( 0 \neq c \in S \) such that \( xsc = cxs \neq 0 \), and \( l_S(\text{Ker}(xsc)) = Sxsc \). Let \( n \in \text{Ker}(xsc), xsc(n) = 0, sc(n) \in \text{Ker}(x) \cap mR = 0, n \in \text{Ker}(sc), \text{Ker}(xsc) \subseteq \text{Ker}(sc) \), so \( \text{Ker}(xsc) = \text{Ker}(sc) \), \( sc \in l_S(\text{Ker}(sc)) = l_S(\text{Ker}(xsc)) = Sxsc \), \( sc = yxsc, y \in S \), \( (1-xy)sc = 0 \). Now \( x \in J(S) \), so \( 1-xy \) is invertible, thus \( sc = 0 \), a contradiction, hence \( x \in \Delta \).

Conversely, let \( z \in \Delta \). Then for each \( c \in S \), \( \text{Ker}(cz) \) is essential in \( M \). Clearly, \( \text{Ker}(cz) \cap \text{Ker}(1-cz) = 0 \), so \( \text{Ker}(1-cz) = 0 \). Also \((1-cz)^2 = (1-cz)(1-cz) = 1-d \) for some \( d \in \Delta \). Since \( d \in \Delta \), \( 0 = \text{Ker}(1-d) = \text{Ker}((1-cz)^2) \). Similarly, we have \( \text{Ker}((1-cz)^n) = 0 \) for all \( n \in Z^+ \). Thus for some \( n \), there exists \( 0 \neq s \in S \) such that \((1-cz)^n s = s(1-cz)^n \neq 0 \), and \( l_S(\text{Ker}(s(1-cz)^n)) = Ss(1-cz)^n \). Since \( \text{Ker}((1-cz)^n) \subseteq \text{Ker}(s(1-cz)^n) \), \( l_S(\text{Ker}(s(1-cz)^n)) \subseteq l_S(\text{Ker}(1-cz)^n) = S \) By QFGP-injectivity, there exists \( a \neq 0 \) such that \( a(1-cz) = (1-cz)a \neq 0 \), and \( l_S(\text{Ker}(a(1-cz))) = Sa(1-cz) \). Now \( cz \in \Delta \), so \( acz \in \Delta \), and \( \text{Ker}(acz) \cap \text{Ker}(a-acz) = 0 \), thus \( \text{Ker}(a-acz) = 0 \). Hence \( S = Sa(1-cz) \), this show that \( 1-cz \) is left invertible for each \( c \in S \). Hence \( z \in J(S) \).

**Corollary 2.13** Let \( R \) be a right FGP-injective ring, then \( J(R) = Z(R_R) \).

A kernel \( Kera \) is called a maximal kernel of \( M \), if for any \( s \in S \), \( Kera \subseteq \text{Kers} \) implies that \( Kera = Kers \) or \( Kers = M \).

**Theorem 2.14** Let \( M_R \) be a right QFGP-injective module which is a self-generator, if \( S \) is semiprime, then every maximal kernel of \( M \) is generated by an idempotent.

**Proof** Let \( L \) be a maximal kernel of \( M \), then there exists \( 0 \neq a \in S \) such that \( L = Kera \).

First we have

\((*)\) \( Kera = Kery \) for every \( 0 \neq y \in Sa \).

Next we shall show that \( L \) is generated by an idempotent. Let \( S_0 = \Delta \cap Sa \). We can proof \( S_0 = 0 \) by the same similar method as \([7,\text{Theorem 7}]\). Consequently, \( a \notin \Delta \), and hence \( Kera \) is not essential in \( M \). Then there exists a nonzero right submodule \( I \) of \( M \) such that \( \text{Ker} \oplus I \) is essential in \( M \). Since \( M \) is a self-generator, take \( 0 \neq m \in I \) such that \( mR = \sum_{f \in T} f(M) \), where \( T \subseteq S \), thus there exists \( b \in T \) such that \( ab \neq 0 \), and hence there exists \( 0 \neq c \in S \) such that \( abc = cab \neq 0 \) and \( l_S(\text{Ker}(abc)) = Sab \). As \( Kera \cap I = 0 \), \( \text{Ker}(abc) = \text{Ker}(bc) \), then \( bc \in l_S(\text{Ker}(bc)) = l_S(\text{Ker}(abc)) = Sab \), thus there exists \( d \in S \)
such that \(bc = dabc\), hence \(bc(M) \in \text{Ker}(a - ada) = 0\), but \(abc(M) \neq 0\), so \(\text{Ker}a \subset \text{Ker}(a - ada)\). Since \(a - ada \in Sa\), then by (\(\ast\)), \(a = ada\). Take \(e = da\), then \(e^2 = e\), and \(L = \text{Ker}a = \text{Ker}e = (1 - e)(M)\).

**Lemma 2.15** If \(M/\text{Soc}(M)\) satisfies ACC on \(M\)–annihilator submodules, then \(\Delta\) is nilpotent.

**Proof** By [8, Theorem 6], it is clear.

Combining Lemma 2.15 and Theorem 2.12, we obtain the following theorem:

**Theorem 2.16** Let \(M\) be a QFGP-injective module. If \(M/\text{Soc}(M)\) satisfies ACC on \(M\)–annihilator submodules, then \(J(S)\) is nilpotent.

As a special case of Theorem 2.14, we obtain the theorem.

**Theorem 2.17** If \(R\) is a right FGP-injective module and \(R/\text{Soc}(R_R)\) satisfies the ACC on right annihilators, then \(J(R)\) is nilpotent.

In [2], An element \(u \in S = \text{End}(M_R)\) is called a right uniform element of \(S\) if \(u \neq 0\) and \(u(M)\) is a uniform submodule of \(M\). An element \(u \in R\) is called right uniform if \(uR\) is a uniform right ideal.

**Theorem 2.18** Let \(M_R\) be a QFGP-injective module and \(S\) its endomorphism ring. Then for any right uniform element of \(S\), the set \(A_u = \{s \in S | \text{Ker}(s) \cap u(M) \neq 0\}\) is a maximal left ideal of \(S\) containing \(l_S(u(M))\).

**Proof** Since \(u(M)\) is uniform, \(A_u\) is a left ideal. It is easy to see that \(l_S(u(M)) \subseteq A_u\) and \(A_u \neq S\), since \(1 \not\in A_u\). Next we shall show that \(A_u\) is maximal. If \(a \not\in A_u\), \(u(M) \cap \text{Ker}a = 0\), whence \(au \neq 0\). Thus there exists \(0 \neq c \in S\) such that \(auc = cau \neq 0\), and \(l_S(\text{Ker}(auc)) = Sauc\). We claim that \(\text{Ker}(auc) = \text{Ker}(uc)\). In fact \(\text{Ker}(uc) \subseteq \text{Ker}(auc)\) is clear. Assume that \(m \in \text{Ker}(auc)\), then \(auc(m) = 0\), whence \(uc(m) \in \text{Ker}(u(M)) = 0\). So \(m \in \text{Ker}(uc)\). Therefore \(uc \in l_S(\text{Ker}(uc)) = l_S(\text{Ker}(auc)) = Sauc\), thus there exists \(y \in S\) such that \((1 - ya)uc = 0\), \(1 - ya \in l_S(uc)\), and \(S = Sa + l_S(uc)\). On the other hand, if \(z \in l_S(uc)\), then \(zuc = 0\), \(0 \neq uc(M) \in \text{Ker}z \cap u(M)\), which implies that \(z \in A_u\), and so \(S = Sa + A_u\). This shows that \(A_u\) is maximal.

**Corollary 2.19** Let \(R\) be right FGP-injective. If \(u \in R\) is a right uniform element, define \(M_u = \{x \in R | r(x) \cap uR \neq 0\}\). Then \(M_u\) is a maximal left ideal which contains \(l(u)\).

**Corollary 2.20** Let \(M_R\) be a self-generator, right QFGP-injective module, if \(M\) is right uniform, then \(S\) is local.

**Proof** By hypothesis, \(J(S) = \Delta = \{x \in S | \text{Ker}x\) is essential in \(M\} = \{x \in S | \text{Ker}x \neq 0\} = \{x \in S | 1(M) \cap \text{Ker}x\} = A_1\). So \(S\) is local.

**Theorem 2.21** Let \(M_R\) be a QFGP-injective module and \(W = \oplus_{i=1}^{n} u_i(M)\) a direct sum of uniform submodule \(u_i(M)\) of \(M\). If \(A \subseteq S\) is a maximal left ideal which is not of the form \(A_u\) for some right uniform element \(u\) of \(S\), then there is \(\psi \in A\) such that \(\text{Ker}(1 - \psi) \cap W\) is essential in \(W\).

**Proof** Let \(k \in A \setminus A_u\), then \(ku \neq 0\), so there exist \(c_1 \in S\) such that \(ku_1c_1 = c_1u_1k \neq 0\), and \(l_S(\text{Ker}(ku_1c_1)) = Sku_1c_1\). Since \(\text{Ker}(ku_1c_1) \subseteq
Ker($u_1c_1$), thus $u_1c_1 \in l_S(\text{Ker}(u_1c_1)) \subseteq l_S(\text{Ker}(kuc_1)) = Sku_1c_1$. Consequently, we have $u_1c_1 = \alpha_1kuc_1$ for some $\alpha_1 \in S$. Let $\varphi_1 = \alpha_1k \in SA \subset A$. Then $(1 - \varphi_1)u_1c_1 = 0$. This shows that $0 \neq u_1c_1(M) \subseteq u_1(M) \cap \text{Ker}(1 - \varphi_1)$. If $\text{Ker}(1 - \varphi_1) \cap u_i(M) \neq 0$ for all $i \geq 2$, then we are done since each $u_i(M)$ is uniform. Then we can prove the theorem by the same method as [2, Lemma 3].

**Theorem 2.22** Let $M$ be a QFGP-injective module which is a self-generator and has finite Goldie dimension.

(1) If $I \subseteq S$ is a maximal left ideal, then $I = Au$ for some right uniform element $u \in S$.

(2) $S$ is semilocal, i.e., $S/J(S)$ is semisimple.

**Proof** This can be proved in the same way as in the proof of [2, Theorem 4].

**Corollary 2.23** Let $R$ be right FGP-injective and right finite dimension.

(1) If $M \subseteq R$ is a maximal left ideal, then $M = Mu$ for some right uniform element $u$ of $R$.

(2) $R$ is semilocal, i.e., $R/J(R)$ is semisimple.

**Theorem 2.24** If $M_R$ is a finitely generated QFGP-injective Kasch module with $S = \text{End}(M_R)$, then

(1) $l_S(\text{Rad}M)$ is essential in $S$.$S$.

(2) $\text{Soc}(S)S$ is essential in $S$.$S$.

(3) For any $s \in S$, $S$s is a minimal left ideal of $S$ if and only if $s(M)$ is a minimal submodule of $M$.

**Proof** If $0 \neq s \in S$, then there exists $0 \neq t \in S$ such that $0 \neq st = ts$ and any $R$–homomorphism from $st(M)$ to $M$ extends to an endomorphism of $M$. Choose a maximal submodule $T$ of the right $R$–module $st(M)$. Since $M$ is right Kasch, there exists a monomorphism $f : st(M) \rightarrow M$. Define $g : st(M) \rightarrow M$ by $g(x) = f(x + T)$. As $M$ is QFGP-injective, $g = u|_{st(M)}$ for some $u \in S$. Take $y \in M$ such that $st(y) \notin T$. Then $u(st(y)) = g(st(y)) = f(st(y) + T) \neq 0$, and thus $ust \neq 0$. If $st(\text{Rad}M) \subsetneq T$, then $st(\text{Rad}M) + T = M$. But $st(M)$ is superfluous in $M$ because $M$ is finitely generated, so $T = st(M)$, a contradiction. Hence $st(\text{Rad}M) \subseteq T$. Thus $ust(\text{Rad}M) = g(st(\text{Rad}M)) = f(0) = 0$, whence $0 \neq st \in Sst \cap l_S(\text{Rad}M)$. This implies that $l_S(\text{Rad}M)$ is essential in $S$.$S$.

(2) Let $0 \neq s \in S$. Since $M_R$ is QFGP-injective, there exists $0 \neq t \in S$ such that $0 \neq st = ts$ and $l_S(\text{Ker}(st)) = Sst$. Let $\text{Ker}(st) \subseteq T$ for some maximal submodule $T$ of $M$, then $l_S(T) \subseteq l_S(\text{Ker}(st)) = Sst \subseteq Ss$. But $l_S(T)$ is minimal by Theorem 2.9, so $\text{Soc}(S)S \cap Ss \neq 0$, and hence $\text{Soc}(S)S$ is essential in $S$.$S$.

(3) If $Ss$ is minimal, then $\text{Kers}$ is maximal by Theorem 2.9, so $s(M) \cong M/\text{Kers}$ is minimal. By Theorem 2.5, the conclusion can be proved.
References


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