A Common Fixed Point Theorem in Menger Spaces

Using Implicit Relation

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Abstract. In an attempt to modify Theorem 1 of [7], which was shown to be not valid in [4], the notion of strict Menger spaces was introduced. In this paper we make use of this notion of a strict Menger space to prove a common fixed point Theorem for self maps. We also obtain a corollary, a fixed point result for six maps.

Keywords: Common fixed point, Compatible maps, Strict Menger space

Mathematical Subject Classification: 47H10, 54H25

1. INTRODUCTION

In this section we give some definitions and results which we use in the next section.

Definition 1.1: [5] A function F: ℝ → [0, 1] is called a distribution function if it is non-decreasing, left continuous, \( \inf_{x \in \mathbb{R}} F(x) = 0 \) and \( \sup_{x \in \mathbb{R}} F(x) = 1 \).

Definition 1.2: [5] A triangular norm \( *: [0,1] \times [0,1] \to [0,1] \) is a function satisfying the following conditions
A triangular norm is also denoted by t-norm.

For any \( a, b \in [0,1] \), if we define

\[
(a * b) = \min(a, b)
\]

then \(*\) is a t-norm and is denoted by ‘min’. We observe that if \( a * b = \min(a, b) \) \( \forall a, b \in [0,1] \), then \(*\) is min t-norm.

**Definition 1.3:** [5] Let \( X \) be a non-empty set and let \( F: X \times X \to \mathbb{D} \) (The set of distribution functions). For \( p, q \in X \), we denote the image of the pair \((p, q)\) by \( F_{p, q} \) which is a distribution function so that \( F_{p, q}(x) = 1 \), for every real \( x \).

Suppose \( F \) satisfies:

a) \( F_{p, q}(0) = 0 \) if and only if \( p = q \)

b) \( F_{p, q}(0) = 0 \)

c) \( F_{p, q}(x) = F_{q, p}(x) \) \( \forall p, q \in X \)

d) If \( F_{p, q}(x) = 1 \) and \( F_{q, r}(y) = 1 \) then \( F_{p, r}(x + y) = 1 \) where \( p, q, r \in X \).

Then \((X, F)\) is called a probabilistic metric space.

**Definition 1.4:** [2] Let \( X \) be a non empty set, \(*\) be a t-norm and \( F: X \times X \to \mathbb{D} \) be a function satisfying

(i) \( F_{p, q}(0) = 0 \) \( \forall p, q \in X \)

(ii) \( F_{p, q}(x) = 1 \) for all \( x > 0 \) if and only if \( p = q \)

(iii) \( F_{p, q}(x) = F_{q, p}(x) \) \( \forall p, q \in X \)

(iv) \( F_{p, r}(x + y) \geq F_{p, q}(x) * F_{q, r}(y) \) for all \( x, y \geq 0 \) and \( p, q, r \in X \).

Then the triplet \((X, F, *)\) is called a Menger space.

**Definition 1.5:** [4] Let \((X, F, *)\) be a Menger space such that \( F_{x,y}(t) \) is strictly increasing in \( t \) when \( x \neq y \). Then \((X, F, *)\) is called a strict Menger space.

**Example 1.6:** Let \((X, d)\) be a metric space. Define \( F_{x,y}(t) = \frac{t}{t + d(x,y)} \) \( \forall t > 0 \) and \( x, y \in X \). If t-norm \(*\) is \( a * b = \min\{a, b\} \) \( \forall a, b \in [0,1] \), then \((X, F, *)\) is a strict Menger space.

**Definition 1.7:** [6]

(i) Let \((X, F, *)\) be a Menger space and \( p \in X \).

For \( \varepsilon > 0, 0 < \lambda < 1 \), the \((\varepsilon, \lambda)\)-neighborhood of \( p \) is defined as \( U_p(\varepsilon, \lambda) = \{ q \in X: F_{p, q}(\varepsilon) > 1 - \lambda \} \). It may be observed that, if \(*\) is continuous then the topology induced by the family \( \{U_p(\varepsilon, \lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1 \} \) is a Hausdorff topology on \( X \) and is known as the \((\varepsilon, \lambda)\) - topology.

(ii) A sequence \( \{x_n\} \) in \( X \) is said to converge to \( p \in X \) in the \((\varepsilon, \lambda)\)-topology, if for any \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\varepsilon, \lambda) \) such that \( F_{x_n, p}(\varepsilon) > 1 - \lambda \) where \( n > N \).
A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence in the \((\varepsilon, \lambda)\)-topology, if for \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\varepsilon, \lambda) \) such that \( F_{x_m, x_n}(\varepsilon) > 1 - \lambda \) for all \( m, n > N \).

A Menger space \((X, F, *)\) where \(*\) is continuous, is said to be complete if every Cauchy sequence in \( X \) is convergent in \((\varepsilon, \lambda)\)-topology.

**Definition 1.8:** [1] Let \( * \) be a t-norm. For any \( \alpha \in [0, 1] \), write \( *_0(\alpha) = 1 \) and \( *_1(\alpha) = *_0(\alpha, \alpha) = *_1(1, \alpha) = \alpha \). In general define \( *_{n+1}(\alpha) = *_n(*_n(\alpha), \alpha) \) for \( n = 0, 1, 2 \ldots \).

Suppose that given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( x > 1 - \delta \) implies \( *_n(x) > 1 - \varepsilon \) \( \forall n \in N \).

Then the sequence \( \{*_n\} \) is said to be equicontinuous at 1. If \( \{*_n\} \) is equicontinuous at 1, then we say that \( * \) is a Hadzic type t-norm.

**Definitions 1.9:** [7] Two self mappings \( A \) and \( B \) of a Menger space \((X, F, *)\) are said to be (i) compatible of type (P) if
\[
F_{ABx_n, BBx_n}(t) \to 1 \quad \text{and} \quad F_{BAX_n, AAX_n}(t) \to 1 \quad \text{for all} \ t > 0
\]
Where \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Bx_n \to z \) for some \( z \) in \( X \) as \( n \to \infty \).

(ii) compatible of type (P1) if
\[
F_{ABx_n, BBx_n}(t) \to 1 \quad \text{for all} \ t > 0
\]
Where \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Bx_n \to z \) for some \( z \) in \( X \) as \( n \to \infty \).

(iii) compatible of type (P2) if
\[
F_{BAX_n, AAX_n}(t) \to 1 \quad \text{for all} \ t > 0
\]
Where \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Bx_n \to z \) for some \( z \) in \( X \) as \( n \to \infty \).

**Lemma 1.10:** [3] Let \((X, F, *)\) be a sequence in a Menger space \((X, F, *)\) with Hadzic-type t-norm \(*\) and \( 0 < \alpha < 1 \). Suppose \( \{x_n\} \) is a sequence in \( X \) such that for any \( s > 0 \), \( F_{x_m, x_{n+1}}(s) \geq F_{x_0, x_1}(\frac{s}{\alpha^n}) \), then \( \{x_n\} \) is a Cauchy sequence.

We observe that ‘min’ t-norm is of Hadzic type.

**Lemma 1.11:** [8] Let \((X, F, *)\) be a Menger space. If there exists \( k \in (0, 1) \) such that \( F_{x,y}(kt) \geq F_{x,y}(t) \) for all \( x, y \in X \) and \( t > 0 \), then \( x = y \).

## 2. Main results

In this section, we make use of the notion of a strict Menger space introduced in [4] to prove a fixed point Theorem for four maps. In this connection, it may not be out of place to mention that the notion of a strict Menger space was introduced to modify, Theorem 1 of [7].

**Theorem 2.1:** Let \( P, Q, R \) and \( C \) be self mappings of a complete strict Menger space
(X, F, *) with t-norm * such that \( t \ast t \geq t \forall t \in [0,1] \), satisfying:

(a) \( P(X) \subseteq R(X), Q(X) \subseteq C(X) \)

(b) there exists a constant \( k \in (0, \frac{1}{2}) \) such that

\[
F_{P,X,Y}(kt) \geq \min \{ F_{P,X,Y}(t) \ast F_{C,X,Y}(t + 0), F_{P,X,Y}(t + 0) \ast F_{C,X,Y}(t) \}
\]

for all \( x, y \in X, t > 0 \)

(c) either \( P \) or \( C \) is continuous

(d) the pairs \( (P, C) \) and \( (Q, R) \) are both compatible of type \( (P_2) \) or type \( (P_2) \)

Then \( P, Q, R \) and \( C \) have a unique common fixed point.

**Proof:** Let \( x_0 \in X \). By (a), there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\begin{align*}
X_{n+1} &= R X_n \quad \text{and} \\
Q X_{n+1} &= C X_n \quad \text{for } n = 0, 1, 2, \ldots
\end{align*}
\]

By taking \( x = x_{2n}, y = x_{2n+1} \) for all \( t > 0 \) in (b), we get

\[
F_{P,X,Y}(kt) \geq \min \{ F_{P,X,Y}(t) \ast F_{C,X,Y}(t + 0), F_{P,X,Y}(t + 0) \ast F_{C,X,Y}(t) \}
\]

for all \( x, y \in X, t > 0 \)

Therefore \( F_{Y_{2n-1},Y_{2n+1}}(kt) \geq F_{Y_{2n-1},Y_{2n+1}}(\frac{t}{2}) \)

Similarly, we can prove that \( F_{Y_{2n+1},Y_{2n+2}}(kt) \geq F_{Y_{2n+1},Y_{2n+2}}(\frac{t}{2}) \)

Hence \( F_{Y_{n+1},Y_n}(t) \geq F_{Y_{n+1},Y_n}(\frac{t}{2}) \forall t > 0, n \in N \)

i.e. \( F_{Y_{n+1},Y_n}(t) \geq F_{Y_{n+1},Y_n}(\frac{t}{2^k}) \geq \cdots \geq F_{Y_0,Y_1}(\frac{t}{(2k)^n}) \)

By Lemma 1.10, \( \{y_n\} \) is a Cauchy sequence.

Since \( (X, F, *) \) is complete, it converges to a point \( z \) in \( X \). Also its sub sequences \( \{P X_{2n}\} \to z, \{C X_{2n}\} \to z, \{Q X_{2n+1}\} \to z \) and \( \{R X_{2n+1}\} \to z \)

Case (i): \( C \) is continuous, \( (P, C) \) and \( (Q, R) \) are compatible of type \( (P_2) \)

CC \( X_{2n} \to Cz, CP X_{2n} \to Cz \) (\( C \) is continuous)

PP \( X_{2n} \to Cz \) (\( P, C \) is compatible of type \( (P_2) \))

By taking \( x = P X_{2n}, y = x_{2n+1} \) in (b), we get \( Cz = z \).

Similarly by taking \( x = z, y = x_{2n+1} \) in (b), we get \( Pz = z \).

Since \( P(X) \subseteq R(X) \), there exists \( w \in X \) such that \( z = Pz = Rw \)

By taking \( x = x_{2n}, y = w \) in (b), we get \( Qw = z \)

Therefore \( Rz = Qz \).

Now by taking \( x = x_{2n}, y = z \) in (b), we get \( Qz = z \).

\( \therefore Pz = Qz = Cz = Rz = z \).
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i.e. $z$ is a common fixed point for $P$, $Q$, $R$ and $C$.

Case (ii): $P$ is continuous and $(P, C), (Q, R)$ are both compatible of type $(P_2)$

$PPx_{2n} \rightarrow Pz, PCx_{2n} \rightarrow Pz$ ($\because P$ is continuous)

$CPx_{2n} \rightarrow Pz$ ($\because (P, C)$ is compatible of type $(P_2)$)

By taking $x = Px_{2n}, y = x_{2n+1}$ in (b), we get

$$F_{PPx_{2n},Qx_{2n+1}}(kt) \geq \min \{F_{PPx_{2n},Rx_{2n+1}}(t) + F_{CPx_{2n},Qx_{2n+1}}(t + 0), F_{PPx_{2n},Rx_{2n+1}}(t + 0) + F_{CPx_{2n},Qx_{2n+1}}(t)\}$$

On letting $n \rightarrow \infty$

$$F_{Pz,z}(kt) \geq \min \{F_{Pz,z}(t) * F_{Pz,z}(t + 0), F_{Pz,z}(t + 0) * F_{Pz,z}(t)\}$$

$$\geq F_{Pz,z}(t)$$

Thus by Lemma 1.11, $Pz = z$.

We have $z = Qz = Cz = Pz$.

$\therefore z$ is a common fixed point for $P$, $Q$, $R$ and $C$.

$\therefore z$ is a common fixed point for $P$, $Q$, $R$ and $C$ when $C$ is continuous (or $P$ is continuous) and $(P, C), (Q, R)$ are compatible of type $P_2$ (or $P_1$)

For uniqueness $v$ be common fixed point for $P$, $Q$, $R$ and $C$.

Take $x = z = Qz = Cz = Pz$.

$i.e. z$ is a common fixed point for $P$, $Q$, $R$ and $C$.

$$F_{Pz,Qv}(kt) \geq \min \{F_{Pz,Rv}(t) * F_{Cz,Qv}(t + 0), F_{Pz,Rv}(t + 0) * F_{Cz,Qv}(t)\}$$

$$F_{z,v}(kt) \geq \min \{F_{z,v}(t) * F_{z,v}(t + 0), F_{z,v}(t + 0) * F_{z,v}(t)\}$$

$$\geq F_{z,v}(t)$$

Thus by Lemma 1.11, $v = z$.

**Corollary 2.2:** Let $A, B, P, Q, S$ and $T$ be self mappings of a complete strict Menger space $(X, F, \ast)$ with continuous $\ast$ such that $t * t \geq t \forall t \in [0,1]$, satisfying:

(a) $P(X) \subseteq ST(X), Q(X) \subseteq AB(X)$

(b) there exists a constant $k \in (0, \frac{1}{2})$ such that

$$F_{Pz,Qy}(kt) \geq \min \{F_{Pz,Ay}(t) * F_{STz,Qy}(t + 0), F_{Pz,Ay}(t + 0) * F_{STz,Qy}(t)\}$$

for all $x, y \in X, t > 0$

(c) either $P$ or $ST$ is continuous

(d) the pairs $(P, ST)$ and $(Q, AB)$ are both compatible of type $(P_1)$ or type $(P_2)$

(e) $AB = BA, ST = TS, PB = BP, QT = TQ$

Then $A, B, P, Q, S$ and $T$ have a unique common fixed point.

**Proof:** Write $C = ST$ and $R = AB$

Then, by Theorem 2.1, there exists $z \in X$ such that $z = Pz = Rz = Qz = Cz$.

Hence $z = Pz = Rz = STz = Qz = Cz = ABz$.

Now $STz = z = T(STz) = Tz = TSTz = Tz = STTz = TZ$
\( \therefore Tz \) is a fixed point for ST.
Since \( ABz = z \Rightarrow BABz = Bz \Rightarrow ABBz = Bz \)
i.e. \( Bz \) is a fixed point for \( AB \).
Similarly, \( ABz = z \Rightarrow AABz = Az \Rightarrow ABAz = Az \)
i.e. \( Az \) is a fixed point for \( AB \).
Therefore \( Az \) and \( Bz \) are fixed points for \( AB \).
Now \( Pz = z \Rightarrow BPz = Bz \Rightarrow PBz = Bz \)
i.e. \( Bz \) is a fixed point for \( P \).
Since \( Qz = z \Rightarrow TQz = Tz \Rightarrow QTz = Tz \)
i.e. \( Tz \) is a fixed point for \( Q \).
Now we prove that \( Bz = Tz \).
By taking \( x = Bz, y = Tz \) in (b), we get
\[
F_{P,Bz,QTz}(kt) \geq \min\{F_{F_{P,Bz,ABTz}}(t) \cdot F_{ST,Bz,QTz}(t + 0),
F_{P,Bz,ABTz}(t + 0) \cdot F_{ST,Bz,QTz}(t)\} \\
\geq \min\{F_{F_{Bz,Tz}}(t) \cdot F_{Bz,Tz}(t + 0),F_{Bz,Tz}(t + 0) \cdot F_{Bz,Tz}(t)\}
\geq F_{Bz,Tz}(t)
\]
Thus by Lemma 1.11, we get \( Bz = Tz \)
\( \therefore Bz \) is a common fixed point for \( P, Q, AB, ST \).
By Theorem 2.1, \( Bz = z = Tz \) is a common fixed point for \( P, Q, AB, ST \).
Since \( ABz = z \Rightarrow Az = z \) and \( STz = z \Rightarrow Sz = z \)
\( \therefore z \) is a common fixed point for \( A, B, S, T, P \) and \( Q \).
For uniqueness, let \( v \) be a common fixed point for \( A, B, S, T, P \) and \( Q \).
By taking \( x = z, y = v \) in condition (b), we get \( z = v \).
\( \therefore z \) is a unique common fixed point for \( A, B, S, T, P \) and \( Q \).

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Received: April, 2010