An EOQ Model for Weibull Deteriorating Items with Power Demand and Partial Backlogging

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Abstract

In this paper an EOQ model is developed for Weibull deteriorating item with power demand pattern in which shortages are allowed and partially backlogged. The backlogging rate is taken to be inversely proportional to the waiting time for the next replenishment. In this model we have considered Weibull two parameter deterioration and power demand pattern. At the end a numerical example is provided to illustrate the problem and sensitivity analysis have been carried out for showing the effect of variation in the parameters.

Mathematics Subject Classification: 90B05

Keywords: Power Demand Pattern, Weibull Deterioration, Partial Backlogging, Shortages

1. Introduction

Deterioration plays an important role in many inventory systems. Deterioration is defined as decay or damage in the quality of the inventory. In some substances like foods, drugs, pharmaceuticals, and radio active substances deterioration takes place during the normal storage period of the units and consequently this loss must be taken
into account when analyzing the system. When the items of the commodity are kept in stock as an inventory for fulfilling the future demand there may be the deterioration of items takes place in the inventory system. Researchers in the field of inventory control have suggested various models taking into consideration different demands and deteriorations. Datta and Pal [15] investigated an inventory system with power demand pattern for items with variable rate of deterioration. Inventory model of deteriorating items with time proportional backlogging rate have been studied by Chang [3], Chang [2] and Dye [1]. Joint pricing and replenishment policy for deteriorating inventory was studied by Wee [5] and Yang [4]. Stock dependent and partial backlogging in inventory model was studied by Wu and Ouyang [8], Teng and Yang [6], and Singh [13]. When the shortage occurs, some customers are willing to wait for back order and others would turn to buy from other sellers. Park [7] and Wang [12] studied shortages and partial back logging of items. Permissible delay in payments was studied by Singh [14], Balkhi and Benkherouf [16]. Finite rate of replenishment was studied by Roychoudhury [9]. Optimal pricing and lot sizing under conditions of perishability and partial back logging was developed by Abad [10]. Hollier and Mak [11] have developed an inventory replenishment policy for deteriorating items in a declining market.

In the present work a deterministic inventory model with power demand pattern is developed in which the deterioration is a Weibull two parameter distribution. Shortages are allowed and partially backlogged in this model. We have shown the effect due to changes in various parameters by taking a suitable numerical example and a sensitivity analysis.

2. Assumption and Notation

In this paper the following assumptions and notations have been used.

i. \( I(t) \) is the inventory level at any time \( t \), \( t \geq 0 \) and \( S \) is the initial inventory level.

ii. \( \mu \) is the life time of items and \( \theta(t) = \alpha \beta t^{\beta-1} \) is the Weibull two parameter deterioration. Where \( 0 < \alpha < 1, \beta > 0 \) are called scale and shape parameter respectively.

iii. \( D(t) \) is the demand rate at any time \( t \) such that \( D(t) = \frac{d t^{1-n}}{n T^{-n}} \), where \( d \) is a positive constant, \( n \) may be any positive number, \( T \) is the planning horizon.
iv. $C, C_1, C_2, C_3$ and $C_4$ denote the set up cost, inventory carrying cost, deterioration cost per unit time, shortage cost for backlogged items and the unit cost of lost sales respectively. All of the cost parameters are positive constants.

v. There is no replenishment or repair of deteriorated items takes place in a given cycle.

vi. The lead time is zero.

vii. A single item is considered over the fixed period $T$ which is subject to weibull deterioration rate.

viii. Deterioration takes place after the life time of items.

ix. The replenishment takes place at an infinite rate.

tax. Shortages are allowed and backlogging rate is 
$$\frac{1-a}{1+a(T-t)} \left(\frac{d t^n}{(n T^n)}\right)$$
when inventory is in shortage. The backlogging parameter $a$ is positive constant and $0 < a << 1$.

3. Formulation of the Model

At the beginning of each cycle the total amount of inventory produced or purchased is assumed as $Q$. Let the initial inventory be $S$. Due to market demand inventory level gradually decreases during the period $(0, \mu)$ and it becomes zero at time $t_1$. Shortages take place in the period $(t_1, T)$ which is partially backlogged. The related figure-1 of the model is as follows.

![Fig.-1](image-url)
The differential equations governing the inventory level at any time $t$ during the cycle $(0, T)$ are given as follows,

$$\frac{dl(t)}{dt} = -D(t), \quad 0 \leq t \leq \mu$$  \hspace{1cm} (1)

$$\frac{dl(t)}{dt} + \theta(t) I(t) = -D(t), \quad \mu \leq t \leq t_1$$  \hspace{1cm} (2)

$$\frac{dl(t)}{dt} = -\frac{D(t)}{1 + a(T - t)}, \quad t_1 \leq t \leq T$$  \hspace{1cm} (3)

The boundary conditions are,

$$I(0) = S, \quad I(t_1) = 0$$  \hspace{1cm} (4)

The solution of equation (1) is

$$I(t) = S - \frac{1}{T^n} \int_0^t dt, \quad 0 \leq t \leq \mu$$  \hspace{1cm} (5)

Taking the first two terms of the exponential series and then integrating we get the solution of equation (2) as

$$I(t) = \frac{d}{T^{1/n}} \left[ \left( t_1^{1/n} - t^{1/n}_i \right) + \frac{\alpha (t_1^{1/n} t^{1/n}) - t^{1/(n+1)}_i}{1 + n\beta} \right] e^{-\alpha t^{1/n}}$$  \hspace{1cm} (6)

Now taking the first two terms of the exponential series and neglecting the term containing $\alpha^2$ the equation (6) becomes

$$I(t) = \frac{d}{T^{1/n}} \left[ \left( t_1^{1/n} - t^{1/n}_i \right) (1 - \alpha t^{1/n}) + \frac{\alpha (t_1^{1/(n+1)} - t^{1/(n+1)}_i)}{1 + n\beta} \right], \quad \mu \leq t \leq t_1$$  \hspace{1cm} (7)

Similarly the solution of equation (3) is

$$I(t) = \frac{d}{T^{1/n}} \left[ \left( t_1^{1/n} - t^{1/n}_i \right) (1 - aT) + \frac{\alpha (t_1^{1/(n+1)} - t^{1/(n+1)}_i)}{1 + n} \right], \quad t_1 \leq t \leq T$$  \hspace{1cm} (8)

From equation (5) and (7) $S$ can be found out as
An EOQ model for Weibull deteriorating items

\[ I(\mu) = S - \frac{d\mu}{T^{1/n}} + \frac{d}{T^{1/n}} \left[ (t_1^{1/n} - \mu^{1/n})(1 - \alpha \mu^\beta) + \frac{\alpha (t_1^{(1/n)+\beta} - \mu^{(1/n)+\beta})}{1 + n\beta} \right] \]

\[ \Rightarrow S = \frac{d\mu}{T^{1/n}} + \frac{d}{T^{1/n}} \left[ (t_1^{1/n} - \mu^{1/n})(1 - \alpha \mu^\beta) + \frac{\alpha (t_1^{(1/n)+\beta} - \mu^{(1/n)+\beta})}{1 + n\beta} \right], \quad \text{(9)} \]

Using equation (9) in equation (5) we get

\[ I(t) = \frac{d}{T^{1/n}} \left[ \mu^{1/n} - t^{1/n} + (t_1^{1/n} - \mu^{1/n})(1 - \alpha \mu^\beta) + \frac{\alpha (t_1^{(1/n)+\beta} - \mu^{(1/n)+\beta})}{1 + n\beta} \right], \quad 0 \leq t \leq \mu, \quad \text{(10)} \]

During period \((0, T)\) total number of units holding \(I_H\) is

\[ I_H = \int_0^{\mu} I(t) \, dt + \int_\mu^T I(t) \, dt \]

Using equation (10) and equation (7) we get

\[ \Rightarrow I_H = \int_0^{\mu} \frac{d}{T^{1/n}} \left[ \mu^{1/n} - t^{1/n} + (t_1^{1/n} - \mu^{1/n})(1 - \alpha \mu^\beta) + \frac{\alpha (t_1^{(1/n)+\beta} - \mu^{(1/n)+\beta})}{1 + n\beta} \right] dt \\
+ \int_\mu^T \frac{d}{T^{1/n}} \left[ (t_1^{1/n} - t^{1/n})(1 - \alpha t^\beta) + \frac{\alpha (t_1^{(1/n)+\beta} - t^{(1/n)+\beta})}{1 + n\beta} \right] dt \]

Calculating further we get

\[ I_H = \frac{d}{T^{1/n}} \left[ (2 + n\beta^2)\alpha \mu^{(1/n)+\beta+1} + \frac{\alpha \beta t_1^{1/n} \mu^{1/n}}{1 + n\beta} + \frac{t_1^{(1/n)}}{1 + n} + \frac{\alpha t_1^{(1/n)+\beta+1}(\beta + n\beta^2)}{n} \right], \quad \text{(11)} \]

Total amount of deteriorated items \(I_D\), during the period \((0, T)\) is

\[ I_D = \int_0^T \theta(t) I(t) \, dt \]
\[ I_D = \int \frac{d}{T^{1/n}} \left[ \left( t_D^{1/n} - t_D^{1/n} \right) (1 - \alpha t_D^\beta) + \frac{\alpha t_D^{(1/n)+\beta}}{1 + n \beta} - \frac{t_D^{(1/n)+\beta}}{1 + n \beta} \right] dt \]

Integrating this, neglecting the term containing \( \alpha^2 \) or higher degree of it as \( 0 < \alpha \ll 1 \), we get

\[ I_D = \frac{d}{T^{1/n}} \left[ \alpha t_D^{(1/n)+\beta} + \frac{n \alpha \beta \mu t_D^{(1/n)+\beta}}{1 + n \beta} - \alpha t_D^{1/n} \mu^\beta \right] \] (12)

Total amount of shortage units \( I_S \) during the period \((0, T)\) is given as

\[ I_S = \int_0^T \left( \frac{d}{T^{1/n}} \left[ \left( t_D^{1/n} - t_D^{1/n} \right) (1 - a T D) + a t_D^{(1/n)+1} - t_D^{(1/n)+1} \right] \right) dt \]

\[ = \frac{d}{T^{1/n}} \left( (T - a T^2) t_D^{1/n} + t_D^{(1/n)+1} \left( \frac{2 a T - 1}{1 + n} \right) - a t_D^{(1/n)+2} + \frac{2 n^2 a T^{(1/n)+2}}{(1 + n) (1 + 2 n)} - \frac{n T^{(1/n)+1}}{1 + n} \right) \] (13)

Total amount of lost sales \( I_L \) during the period \((0, T)\) is given by

\[ I_L = \int_t \left( 1 - \frac{1}{1 + a (T - t)} \right) D(t) \ dt \]

\[ = \frac{d}{T^{1/n}} \left[ \frac{n T^{(1/n)+1}}{1 + n} + \frac{t_D^{(1/n)+1}}{1 + n} - t_D^{1/n} \right] T \] (14)

Total average cost of the system per unit time is given by

\[ K = \frac{1}{T} \left[ C + C_1 I_H + C_2 I_D + C_3 I_S + C_4 I_L \right] \]

\[ = \frac{C}{T} + \frac{C_1}{T^{(1/n)+1}} \left[ (2 + \beta + n \beta^2) \alpha t_D^{(1/n)+\beta+1} \right] \frac{1 + \beta}{(1 + n \beta)((1/n) + \beta + 1))} - \frac{\alpha \beta t_D^{1/n} \mu^{1+\beta}}{1 + n \beta} + \frac{t_D^{(1/n)+1}}{1 + n} + \frac{\alpha t_D^{(1/n)+\beta+1} (\beta + n \beta^2)}{n} \]

\[ + \frac{C_2}{T^{(1/n)+1}} \left[ \frac{\alpha t_D^{(1/n)+\beta}}{1 + n \beta} + \frac{n \alpha \beta \mu t_D^{(1/n)+\beta}}{1 + n \beta} - \alpha t_D^{1/n} \mu^\beta \right] - \frac{C_3}{T^{(1/n)+1}} \left( T - a T^2 \right) t_D^{1/n} + t_D^{(1/n)+1} \]
An EOQ model for Weibull deteriorating items

\[
\times \left[ \frac{2aT - 1}{1 + n} \right] - \frac{a T^{(1/n)^2}}{1 + 2n} + \frac{2n^2 a T^{(1/n)^2}}{(1 + n)(1 + 2n)} - \frac{n T^{(1/n)^2}}{1 + n} + C_4 \frac{d a}{T^{(1/n)^2}} \left[ \frac{n T^{(1/n)^2}}{1 + n} + \frac{T^{(1/n)^2}}{1 + n} - t_1^{(1/n)^2} \right]
\]

Optimal value of \( t_1 \) can be found out by solving the following equation

\[
\frac{df}{dt_1} = 0
\]

\[
\Rightarrow C_1 \left[ t_1^{(1/n)} + \left( \frac{1}{n} + \beta + 1 \right) \left( n \beta^2 + \beta \right) \alpha t_1^{(1/n) + \beta} - \frac{\alpha \beta t_1^{(1/n) - 1} \mu^{1+\beta}}{1 + \beta} \right] + C_2 \left[ \alpha t_1^{(1/n) + \beta - 1} - \alpha t_1^{(1/n) - 1} \mu^{\beta} \right] - C_3 \left[ (T - a T^2) t_1^{(1/n) - 1} + (2aT - 1) t_1^{(1/n) - 1} \right] + C_4 a \left( t_1^{(1/n)} - T t_1^{(1/n) - 1} \right) = 0
\]

The minimum total average cost per unit time is obtained for those values of \( t_1 \) for which

\[
\frac{d^2 K}{dt_1^2} > 0
\]

By solving equation (16) the value of \( t_1 \) can be obtained and then from equation (9) and (15), the optimal value of \( S \) and \( K \) can be found out respectively.

4. Numerical Example

Let us take

\[
[\alpha, \beta, a, n, d, \mu, T, C, C_1, C_2, C_3, C_4] = [0.1, 2, 0.2, 4, 60, 0.4, 1, 200, 12, 10, 4, 8]
\]

in their respective proper units. Then we get \( t_1 = 0.243328, S = 42.1726, K = 269.375 \)

5. Tabulation and Sensitivity Analysis

We now examine the sensitivity analysis of the optimal solution of the model for changes in \( \alpha, \beta, a \) and \( \mu \) parameter values associated with the system in the following Table 5.1. We change one parameter at a time keeping the other parameters
unchanged. Sensitivity analysis is performed by changing the parameters $\alpha$, $\beta$, $a$ and $\mu$ by -50%, -40%, -30%, -20%, -10%, +10%, +20%, +30%, +40%, 50% one by one in the model that are given in the following table. The initial data of all parameters have been taken from the above numerical example.

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<th>Parameter</th>
<th>% change</th>
<th>$t_1$</th>
<th>$S$</th>
<th>$K$</th>
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From the above table we can conclude the following:
A. Effects of various parameters with the increment of scale parameter $\alpha$
   a) The inventory period decreases.
   b) Initial inventory level also decreases.
   c) Total average cost of the system increases.
B. Effects of various parameters with the increment of shape parameter $\beta$
   a) Variable changes occur in the inventory period.
   b) Initial inventory level also changes variability.
   c) Total average cost of the system decreases.
C. Effects of various parameters with the increment of back logging parameter $\alpha$
   a) There is an increase in inventory period.
   b) Initial inventory level also increases.
   c) Total average cost of the system increases.
D. Effects of various parameters with the increment of life time parameter $\mu$
   a) There is an increase in inventory period.
   b) Initial inventory level also increases.
   c) Total average cost of the system increases.

6. Conclusion

In the present work we have developed an inventory model with power pattern demand with Weibull deterioration rate. Shortages have been allowed and completely backlogged in this model. This type of power pattern demand requires a different policy than the conventional policy based on general Weibull pattern. In cases where large portion of demand occurs at the beginning of the period we use $n > 1$ and when it is large at the end we use, $0 < n < 1$. Similarly $n = 1$ and $n = \infty$ corresponds to constant demand and instantaneous demand respectively. Behaviors of different parameters have been illustrated through the numerical example and sensitivity analysis.

References


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