Generating Functions of Generalized Hyper
Geometric Functions by Group Theoretical
Method

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Abstract

Louis Weisner in [2] introduce Lie operators and then obtained generating functions for generalized hypergeometric corresponding to increasing and decreasing of numerator parameter a. Monoch and Jain [5], Agrawal and Jain [1] etc., have also applied it to obtain generating function by variation of parameter a. This paper is an attempt to exhibit the group theoretic significance of generating relations for generalized hyper geometric function. Weisner's group theoretic method is utilized to obtain generating relations of generalized hypergeometric functions with the introduction of raising operators.

Keywords: hyper geometric functions ; generating functions.

1. Introduction

The generalized hyper geometric polynomial \( p+1 F_q \) \(((a+n); (b_p); ((c+n)q); z\) is defined by \[9\]
\[
p+1 F_q \) \(((a+n); (b_p); ((c+n)q); z\] = \sum_{k=0}^{\infty} \frac{(a+n)_k (b_p)_k z^k}{((c+n)_k k!}
\]
(1.1)

Where \( (b)_n = \frac{\Gamma(b+n)}{\Gamma(b)} \), \( n \in N \), \( (b)_0 = 1 \) and \( (b_p) \) abbreviates the array of \( P \)-parameters \( b_1, \ldots, b_p \), with similar interpretation for \( (c+n)_p \).
$p+1\, F_q[(a+n); (bp); ((c+n)q); z]$ satisfies the following ordinary differential equation [7]

$$(z\frac{d}{dz} + b_i)F = b_iF(b_i +), i = 1,2,\ldots, p$$

(1.2)

Where $p+1\, F_q[(a+n); (bp); ((c+n)q); z]$ and

$F(b_i +) = p+1\, F_q[(a+n),b_1,\ldots,b_{i-1},b_i +1,b_{i+1},\ldots,b_p,(Bq);z]$,

In section 2, the ordinary differential equation (1.2) is used to obtain the first order partial differential raising operators (PDRO). Further we find the extended forms of the transformation groups generated by these operation

2. Derivative of the generating function:

For the hyper geometric polynomial $p+1\, F_q[(a+n); (bp); ((c+n)q); z]$, we consider the following PDRO $A_i$, $B_i$ and $C_i$ ($i=1,2,\ldots,p$) defined by

$$A_i = y_i \frac{\partial}{\partial y_i}$$

(2.1)

$$B_i = y_i^{-1}z(1-z)\frac{\partial}{\partial z} - bz + c - 1 + y_i \frac{\partial}{\partial y_i}$$

(2.2)

$$C_i = y_i (1-z)\frac{\partial}{\partial z} - a + y_i \frac{\partial}{\partial y_i}$$

(2.3)

Such that

$$A_i \left( y_1^{n_1} \ldots y_{i-1}^{n_{i-1}} y_i^n y_{i+1}^{n_{i+1}} \ldots y_p^{n_p} \right)_{p+1} F_q[(a+n); (bp); ((c+n)q); z]$$

(2.4)

$$= n_i \left( y_1^{n_1} \ldots y_{i-1}^{n_{i-1}} y_i^n y_{i+1}^{n_{i+1}} \ldots y_p^{n_p} \right)_{p+1} F_q[(a+n),b_1,\ldots,b_{i-1},b_i +1,b_{i+1},\ldots,b_p,(Bq);z]$$

(2.5)

$$B_i \left( y_1^{n_1} \ldots y_{i-1}^{n_{i-1}} y_i^n y_{i+1}^{n_{i+1}} \ldots y_p^{n_p} \right)_{p+1} F_q[(a+n); (bp); ((c+n)q); z]$$

(2.6)

$$= (a+n_i)(c+n_i-b)\frac{(c+n_i)}{(c+n_i-b)} \left( y_1^{n_1} \ldots y_{i-1}^{n_{i-1}} y_i^n y_{i+1}^{n_{i+1}} \ldots y_p^{n_p} \right)_{p+1} F_q[(a+n + 1),b_1,\ldots,b_{i-1},b_i +1,b_{i+1},\ldots,b_p,(c+n+1)q);z]$$

The commutate relations satisfied by $A_i$, $B_i$ and $C_i$ are

$$[A_i, B_i] = -B_i, [A_i, C_i] = C_i, [B_i, C_i] = b - a - c - 2A_i + 1$$
These commutator relations show that:
{1, A_i, B_i, C_i} form a Lie group, with identity operator 1.
It can be easily shown in the partial differential operator L given by
L = B_i C_i + (a + A_i)(c-b+A_i)
The extended form of the group generated by B_i, C_i, (i=1,2,….,p) are given by
\[
\exp(WB_i)\int f(y_i, z) = (1 - z + ze^{W/y_i})^{-b} e^{(c-1)W/y_i} f(y_i e^{W/y_i}, \frac{ze^{W/y_i}}{1-z + ze^{W/y_i}})
\] (2.7)
And
\[
\exp(\mu C_i)\int f(y_i, z) = e^{\mu - y_i \alpha} f(y_i e^{-\mu y_i \alpha}, 1 - (1 - z)e^{-\mu y_i})
\] (2.8)
(2.6) and (2.7) have been obtained by using Miller's Technique [3]

Generating Functions Annulled by Conjugate of (A_i - n_i)

We see that
\[
u(z, y_i)\int y_i^n_1 y_i^n_2 y_i^n_3 y_i^n_4 \ldots y_i^n_p \int_{n=0}^{p+1} F_q [(a + n); (bp); ((c + n)q); z]
\]
are solution of the simultaneous equation Lu=0 and A_i u = n_i u for arbitrary n_i

Now
\[
\exp(WB_i + \mu C_i)\int y_i^n_1 y_i^n_2 y_i^n_3 y_i^n_4 \ldots y_i^n_p \int_{n=0}^{p+1} F_q [(a + n); (bp); (c + n)q); z]
\] (3.1)
\[
= \left( y_i^n_1 y_i^n_2 y_i^n_3 y_i^n_4 \ldots y_i^n_p \right) \left( 1 - z + ze^{W/y_i} \right)^b e^{c+n_i W/y_i} f(y_i e^{W/y_i}, \frac{ze^{W/y_i}}{1-z + ze^{W/y_i}})
\]
\[
\times_{q=0} F_q [(a + n); (bp); (c + n)q); (1 - (1 - \frac{ze^{W/y_i}}{1-z + ze^{W/y_i}}) e^{-\mu y_i})]
\]
Where |z| ≤ 1 and c is not an integer.

Put S = exp(W B_i + \mu C_i), then SA_i S^{-1} is a conjugate of A_i and C_i (z, y_i) is annulled by L and S(A_i - n_i) S^{-1}.

Now we consider the following cases:

Case 1:- Let \mu = 0 and W = 1, (3.1) reduces to:
\[
\exp(B_i)\int y_i^n_1 y_i^n_2 y_i^n_3 y_i^n_4 \ldots y_i^n_p \int_{n=0}^{p+1} F_q [(a + n); (bp); ((c + n)q); z]
\]
\[
= \left( y_i^n_1 y_i^n_2 y_i^n_3 y_i^n_4 y_i^n_5 \ldots y_i^n_p \right) \left( 1 - z + ze^{W/y_i} \right)^b e^{(c+1+n) W/y_i} f(y_i e^{W/y_i}, \frac{ze^{W/y_i}}{1-z + ze^{W/y_i}})
\]
\[
\times_{q=0} F_q [(a + n); (bp); (c + n)q); (1 - (1 - \frac{ze^{W/y_i}}{1-z + ze^{W/y_i}}) e^{-\mu y_i})]
\]
Also

\[
\exp(B_i)\left[y_1^{n_1} \cdots y_i^{n_i} y_{i+1}^{n_{i+1}} \cdots y_p^{n_p}\right]_{p+1} F_q\left[(a+n);(bp);(c+n)q;z\right]
\]

\[
= \sum_{k=0}^{\infty} \binom{c+n-1}{k} y_1^{n_i-k} \cdots y_i^{n_i-k} y_{i+1}^{n_{i+1}-k} \cdots y_p^{n_p+k} \times_{p+1} F_q\left[(a+n-k);(bp);(c+n-k)q;z\right]
\]

Equating the two values and after appropriate adjustment, we get:

\[
\left(1 - z + ze^\mu\right)^{-b} e^{\left(c-\nu+n\right)\mu} \times_{p+1} F_q\left[(a+n);(bp);(c+n)q;\left(1 - \frac{ze^\mu}{1 - z + ze^\mu}\right)\right]
\]

\[
= \sum_{k=0}^{\infty} \binom{c+n-1}{k} t^k \times_{p+1} F_q\left[(a+n-k);(bp);(c+n-k)q;z\right]
\]

(3.2)

Where \( t = 1/y_i \), in particular, when \( n = 0 \), (3.2) gives

\[
\left(1 - z + ze^\mu\right)^{-b} e^{\left(c-\nu+n\right)\mu} \times_{p+1} F_q\left[(a);(bp);(c)q;\left(1 - \frac{ze^\mu}{1 - z + ze^\mu}\right)\right]
\]

\[
= \sum_{k=0}^{\infty} \binom{c-1}{k} t^k \times_{p+1} F_q\left[(a-k);(bp);((c-k)q);z\right]
\]

Case II: Let \( \mu = 1 \) and \( W = 0 \), (3.1) reduces to:

\[
\exp(C_i)\left[y_1^{n_1} \cdots y_i^{n_i} y_{i+1}^{n_{i+1}} \cdots y_p^{n_p}\right]_{p+1} F_q\left[(a+n);(bp);(c+n)q;z\right]
\]

\[
= \left(y_1^{n_1} \cdots y_i^{n_i} y_{i+1}^{n_{i+1}} \cdots y_p^{n_p}\right) e^{-\gamma(a+n)} \times_{p+1} F_q\left[(a+n);(bp);(c+n)q;\left(1 - (1-z)e^{-\gamma}\right)\right]
\]

Also

\[
\exp(C_i)\left[y_1^{n_1} \cdots y_i^{n_i} y_{i+1}^{n_{i+1}} \cdots y_p^{n_p}\right]_{p+1} F_q\left[(a+n);(bp);(c+n)q;z\right]
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \binom{c+n-k}{k} y_1^{n_i-k} \cdots y_i^{n_i-k} y_{i+1}^{n_{i+1}-k} y_p^{n_p-k} \times_{p+1} F_q\left[(a+n);(bp);(c+n-k)q;z\right]
\]

Equating the two values and after minor adjustment, we get:
Generating functions

\[ e^{y_1(a+n)F_{p+1}[(a+n);(bp);(c+n)q;(1-(1-z)e^{-y_1})] \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( y_1^{a-k} \ldots y_{p+1}^{a-k} \ldots y_p^{a-k} \right) \]

\[ \times \frac{(a+n)_k(c+n-b)_k}{(c+n)} \]

\[ \sum_{p+1} F_q[(a+n+k);(bp);(c+n+k)q;z] \]  

(3.3)

In Particular, when \( n=0 \), (3.3) gives

\[ e^{y_1(a)F_{p+1}[(a);(bp);(c)q;(1-(1-z)e^{-y_1})] \]

\[ = \sum_{k=0}^{\infty} \frac{(-y_1)^k}{k!} \left( (a)_k(c-b)_k \right) \]

\[ \frac{(c)}{p+1} F_q[(a+k);(bp);(c+k)q;z] \]  

Case III: - Let \( \mu \neq 0 \) and \( W=1 \), (3.1) reduces to:

\[ \exp(B_1 + \mu C_1) \left[ y_1^n \ldots y_{p+1}^{n} \ldots y_p^{n} \right] \]

\[ \times \frac{1}{p+1} F_q[(a+n);(bp);(c+n)q;z] \]

\[ = \left( y_1^n \ldots y_{p+1}^{n} \ldots y_p^{n} \right) \left[ 1 - ze^{1/y_1} \right] \]

\[ \times \frac{1}{p+1} F_q[(a+n);(bp);(c+n)q;\left(1-(1-ze^{1/y_1})e^{-y_1}z\right)] \]

(3.4)

Also

\[ \exp(B_1 + \mu C_1) \left[ y_1^n \ldots y_{p+1}^{n} \ldots y_p^{n} \right] \]

\[ \times \frac{1}{p+1} F_q[(a+n);(bp);(c+n)q;z] \]

\[ = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{c+n+k-1}{m} \right) \frac{\mu^k y_1^{m-k}}{k!} \]

\[ \times \frac{(a+n)_k(c+n-b)_k}{(c+n)} \]

Equating both values and after appropriate adjustment, we get:

\[ (1 - ze^{\mu y_1}) e^{(e^{(-1+n)\mu y_1} - 1)(a+n)} e^{\mu y_1} \]

\[ \times \frac{1}{p+1} F_q[(a+n);(bp);(c+n)q;\left(1-(1-ze^{1/y_1})e^{-y_1}z\right)] \]

\[ = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{c+n+k-1}{m} \right) \frac{\mu^k t^{m-k}}{k!} \]

\[ \times \frac{(a+n)_k(c+n-b)_k}{(c+n)} \]

\[ p+1 F_q[(a+n+k-m);(bp);(c+n+k-m)q;z] \]
In particular, when \( n = 0 \) and \( t = 1/y \), (3.4) gives

\[
(1 - z + ze^t) e^{(e^{-1})t - ze^{-x}} x_{p+1} F_q\left( (a); (bp); (c)q; \left(1 - \frac{ze^t}{1 - z + ze^t}\right)e^{-x+e^t}\right)
\]

\[
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\begin{array}{c}
  c + k - 1 \\
  m
\end{array}\right) \mu^k \frac{l!}{k!} \frac{(a)_k}{(c)_k} \frac{(c - b)_k}{(c)_k} x_{p+1} F_q\left( (a + k - m); (bp); ((c + k - m)q); z\right)
\]

References


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