The Causal Riesz Diamond Kernel

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Abstract
The elliptic kernel due to Marcel Riesz has been generalized for the ultrahyperbolic metric by Nozaki (cf. [6]) and by Trione (cf. [8]) for the causal and anticausal cases. This kernels are elementary solutions of the Laplacian and of the ultrahyperbolic differential operators.

In this paper we introduce a family of generalized function that are elementary solution of the Riesz Diamond differential operator. They contain as particular cases the elliptic, hyperbolic and causal Riesz kernels. Also some basic properties are studied.

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I Preliminaries

This article deals with a generalization of the causal (anticausal) Riesz kernel introduced by Trione (cf. [8]) depending on the \((P \pm i0)\) generalized functions by means a new family of distributions depending on the \(P(m, x)\) functions, in particular when \(m = 2\).

To do this we start by remembering some definitions.

Let the Riesz Diamond differential operator \(\diamond\) introduced by Kananthai (cf. [4]) defined as

\[
\diamond = \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left( \frac{\partial^2}{\partial x_{p+1}^2} + \ldots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2
\]

(I.1)

where \(p + q = n\), \(n\) the dimension of the space \(\mathbb{R}^n\), which may be factorized as the “product” of the ultrahyperbolic differential operator and the Laplacian

\[
\diamond = \Box \Delta
\]

(I.2)
where
\[ \Box = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} \] (I.3)

and
\[ \triangle = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \] (I.4)

Based of this “product” in finding elementary solution of the operator $\diamond$ he uses the convolution of functions that are elementary solution of the operators $\Box$ and $\triangle$. Later Aguirre and Kananthai (cf.[1]) defined the Riesz Diamond kernel $K_{\alpha,\beta}(x)$ as

\[ K_{\alpha,\beta}(x) = R_\alpha^e \ast R_\beta^H \] (I.5)

where $R_\alpha^e$ is the elliptic Riesz kernel of orden $\alpha$

\[ R_\alpha^e = \frac{|x|^{\alpha-n}}{H_n(\alpha)} \] (I.6)

where $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$, $\alpha$ is a complex number, $n$ is the dimension of $\mathbb{R}^n$ and the constant $H_n(\alpha)$ is given by

\[ H_n(\alpha) = \frac{\pi^{\frac{n+1}{2}} \Gamma(\alpha) \Gamma(n-\alpha+1)}{\Gamma(n+1)} \]

and $R_\beta^H$ is the ultrahyperbolic Riesz kernel introduced by Nozaki [6, p.72] given by

\[ R_\beta^H = \begin{cases} \frac{u^{\frac{\beta-n}{2}}}{K_n(\beta)} & \text{if } x \in K_+ \\ 0 & \text{if } x \notin K_+ \end{cases} \] (I.7)

where $K_n(\beta)$ is the constant given by

\[ K_n(\beta) = \frac{\pi^{n-1} \Gamma\left(\frac{2+\beta-n}{2}\right) \Gamma\left(\frac{1-\beta}{2}\right) \Gamma(\beta)}{\Gamma\left(\frac{2+\beta-p}{2}\right) \Gamma\left(\frac{p-\beta}{2}\right)} \] (I.8)

$\Gamma(z)$ is the Euler Gamma function, $u$ is the quadratic form in $n$ variables.
The causal Riesz diamond kernel

\[ u = u(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2 \] (I.9)

where \( p + q = n \), \( \beta \) a complex number and \( K_+ \) denotes the interior of the forward cone defined as \( K_+ = \{ x \in \mathbb{R}^n : x_1 > 0, u > 0 \} \); \( \overline{K}_+ \) denotes its closure.

The support of the function \( R^H_\alpha(u) \) is included in the cone \( \overline{K}_+ \).

It may be observed that the elliptic kernel \( R^\alpha_\alpha \) and the ultrahyperbolic kernel \( R^\beta_\beta \) which are ordinary function if \( \text{Re}(\alpha) \geq n \), and \( \text{Re}(\beta) \geq n \), respectively, are distributions for \( \text{Re}(\alpha) < n \) and \( \text{Re}(\beta) < n \).

By putting \( p = 1 \) in (I.7) and (I.8), \( R^H_\alpha \) reduces to the hyperbolic Riesz kernel (cf. [7, p.31]) given by

\[
R^H_\alpha = \begin{cases} 
\frac{u^\alpha}{H_n(\alpha)} & \text{if } x \in K_+ \\
0 & \text{if } x \notin K_+
\end{cases}
\] (I.10)

where \( u = x_1^2 - x_2^2 - \ldots - x_n^2 \) and

\[
H_n(\alpha) = 2^{n-1} \pi^n \frac{\alpha^n}{\pi} \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{\alpha - n + 2}{2} \right)
\]

**Definition 1** Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the Euclidean \( n \)-dimensional space \( \mathbb{R}^n \) and let \( P = P(x) \) be the quadratic form in \( n \) variables given in (I.9) that we rewrite:

\[
P = P(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2.
\] (I.11)

Gelfand and Shilov (cf. [3]) defined the \( (P \pm i0)^\lambda \) generalized function as the limit:

\[
(P \pm i0)^\lambda = \lim_{\epsilon \to 0} (P \pm i\epsilon |x|^2)^\lambda
\] (I.12)

where \( \epsilon \) is a positive real number, \( \lambda \) a complex number and

\[
|x|^2 = x_1^2 + x_2^2 + \ldots + x_n^2.
\]
They are analytic function on \( \lambda \) everywhere except at \( \lambda = -\frac{n}{2} - k, k = 0, 1, 2, \ldots \), where they have simple poles (cf. [3, p. 275]).

Frequently it is resed the following expression of the \((P \pm i0)\lambda\) distribution in term of the \(P_+\) and \(P_-\) generalized functions defined by:

\[
P^\lambda_+ = \begin{cases} 
P^\lambda & \text{if } P > 0 \\
0 & \text{if } P \leq 0 \end{cases} \quad (I.13)
\]

\[
P^\lambda_- = \begin{cases} 
0 & \text{if } P \geq 0 \\
|P|^\lambda & \text{if } P < 0 \end{cases} \quad (I.14)
\]

Then, we have

\[(P \pm i0)^\lambda = P^\lambda_+ + e^{\pm i\pi \lambda} P^\lambda_- \quad (I.15)\]

Let us now consider the causal (anticausal) ultrahyperbolic kernel \(H_\alpha(P \pm i0, n)\) given by the following

**Definition 2** Let \(\alpha\) be a complex number. Trione introduced the causal (anticausal) distributions \(H_\alpha(P \pm i0)\) as follows

\[H_\alpha(P \pm i0, n) = C(\alpha, n)(P \pm i0)^{\alpha-n} \quad (I.16)\]

where

\[C(\alpha, n) = \frac{e^{\pm i\frac{\pi}{2}} \Gamma\left(\frac{n-\alpha}{2}\right)}{2^{\alpha} \Gamma\left(\frac{n}{2}\right) \pi^{\frac{\alpha-n}{2}}} \quad (I.17)\]

Its Fourier transform is

\[\mathcal{F}[H_\alpha(P \pm i0, n)] = (Q \mp i0)^{\frac{\alpha}{2}} \quad (\text{cf.}[8]) \quad (I.18)\]

This generalized function is the causal (anticausal) analogue of the elliptic kernel due to M. Riesz and have analogue properties that we collecte in the following Lemma.

**Lemma 1** Let \(\alpha\) be a complex number and let \(k\) by a non negative integer. Then we have

1. \(H_\alpha * H_{-2k} = H_{\alpha-2k}\)
2. $H_0 = \delta$

3. $H_\alpha \ast H_\beta = H_{\alpha+\beta}$

4. The distributions $H_{2k}$, $2k \neq n + 2r$; $r = 0, 1, 2, \ldots$, are elementary solutions of the homogeneous ultrahyperbolic operator $k$ times

$$L^k \{H_{2k}\} = \delta$$

where

$$L^k = \left\{ \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right\}^k$$

**Proof.** For the Proof we referd to Trione [7].

## II Main results

Let us now consider the following function $\mathcal{P}^\lambda(m, x)$ introduced by Aguirre and Kananthai (cf. [1])

$$\mathcal{P}^\lambda(x, m) = \left( \left( \sum_{i=1}^{p} x_i^2 \right)^m - \left( \sum_{i=p+1}^{p+q} x_i^2 \right)^m \right)^\lambda$$

where $m$ is a positive integer, and $\lambda$ a complex number.

In the particular case for $m = 2$ and $\lambda = 1$, we have the function

$$\mathcal{P}(2, x) = \left( \sum_{i=1}^{p} x_i^2 \right)^2 - \left( \sum_{i=p+1}^{p+q} x_i^2 \right)^2 \quad (II.1)$$

that may by factorized as the following product

$$\mathcal{P}(2, x) = r^2 P \quad (II.2)$$

where $r^2 = (x_1^2 + \ldots + x_n^2)$, and

$$P = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$$

By means of suitable changes of variables according the procedure by Gelfand and Shilov it can be found that $\mathcal{P}^\lambda(2, x)$ has two sets of poles, the first one at $\lambda = -1, -2, \ldots, -k, \ldots$; and the second at $\lambda = -\frac{n}{4}, -\frac{n}{4} - 1, \ldots, -\frac{n}{4} - k, \ldots$; $k$ a non negative integer.
Notece that by change of variables the function $\mathcal{P}(2, x)$ and also $\mathcal{P}(m, x)$ may be expressed as a quadratic form in two variables we may obtain new generalized functions by applying the ideas of Gelfand and Shilov.

Then we have the following

**Definition 3** Let $\lambda$ be a complex number. We define the $(\mathcal{P}(2, x) \pm i0)^\lambda$ generalized function (cf [2]) as the limit

$$
(G \pm i0)^\lambda = (\mathcal{P}(2, x) \pm i0)^\lambda = \lim_{\epsilon \to 0} (r^2 P + i\epsilon r^4)^\lambda = (r^2 P + i0)^\lambda \quad (\text{II.3})
$$

$(G \pm i0)^\lambda$ is analytic in $\lambda$ every where except at $\lambda = -\frac{n}{4} - k$, $k = 0, 1, 2, ...$ where have simple poles.

**Definition 4** Let $\alpha$ be a complex number. We define the generalized function

$$
\mathcal{H}_\alpha(G \pm i0, n) = \frac{e^{\mp i\frac{\pi}{2}q\pi n^2}}{2^{\alpha + \frac{n}{2} \pi i n^2}} (G \pm i0)^{-\alpha - \frac{n}{4}} \quad (\text{II.4})
$$

This distributional function are analogue to the causal (anticausal) ultrahyperbolic kernel given by (I.16) and have analogues properties that we will use to obtain elementary solutions of the Riesz Diamond differential operator given by (I.1) iterated $k$-times.

Taking into account that the Fourier transform of $(G \pm i0)^\lambda$ is (cf. [2])

$$
\mathfrak{F}[(G \pm i0)^\lambda] = \frac{e^{\mp i\frac{\pi}{2}q2\lambda n^2 + n^2 \pi n\Gamma(\frac{\alpha + n}{2})}}{\Gamma(-2\lambda)} (\overline{G} \pm i0)^{-\lambda - \frac{n}{2}} \quad (\text{II.5})
$$

we obtain the Fourier transform of $\mathcal{H}_\alpha(G \pm i0, n)$:

$$
\mathfrak{F}[\mathcal{H}_\alpha(G \pm i0, n)] = \frac{1}{(2\pi)^{\frac{n}{2}}} (\overline{G} \mp i0)^{-\frac{n}{2}} \quad (\text{II.6})
$$

where $\overline{G}$ is the dual of the function $r^2 P$, as usual, (cf. [2]) that we write

$$
\overline{G} = \left( \sum_{i=1}^{n} \xi_i^2 \right) \left( \sum_{i=1}^{p} \xi_i^2 - \sum_{i=p+1}^{p+q} \xi_i^2 \right) \quad (\text{II.7})
$$

Based on the expression of the Fourier transform of $(G \pm i0)^\lambda$ we will prove some Theorems related to the kernel introduced in (II.14) that we named the causal Riesz Diamond kernel.
Interesting is to consider the particular case when $\alpha = -4k$, $k$ a non negative integer.

Formally we have

$$\mathfrak{F}[H_{-4k}(G \pm i0, n)] = \frac{1}{(2\pi)^{\frac{n}{2}}}(G \mp i0)^k$$  \hspace{1cm} (II.8)

Taking into account that when $k$ is a non negative integer we have

$$(G + i0)^k = (G - i0)^k = G^k$$  \hspace{1cm} (II.9)

it results

$$\mathfrak{F}[H_{-4k}(G \pm i0, n)] = \frac{1}{(2\pi)^{\frac{n}{2}}}(G)^k$$  \hspace{1cm} (II.10)

Let us now consider the Riesz Diamond operator iterated $k$-times and its Fourier transform (cf.[5])

$$\mathfrak{F}[\diamond^k \delta] = (-1)^k \frac{1}{(2\pi)^{\frac{n}{2}}}(r^2 P(\xi))^k = (-1)^k \frac{1}{(2\pi)^{\frac{n}{2}}}(\mathcal{G})^k$$  \hspace{1cm} (II.11)

where $\mathcal{G}$ is the same that in (II.7).

From (II.8) and (II.11) and by the uniqueness of the Fourier transform we obtain that

$$H_{-4k} = (-1)^k \diamond^k \delta$$  \hspace{1cm} (II.12)

that expresses that $H_{-4k}$ is a combination of de delta Dirac distribution and its derivatives. Then $H_{-4k}$ is a convolutor on $\mathcal{D}'$.

Moreover when $k = 0$, we have

$$H_0 = \delta$$  \hspace{1cm} (II.13)

This results we collect in the following

**Lemma 2** Let $k$ be a non negative integer.

Then $H_{-4k}(G \pm i0, n)$ is a convolutor on $\mathcal{D}'$ and we have

$$H_{-4k}(G \pm i0, n) = (-1)^k \diamond^k \delta$$  \hspace{1cm} (II.14)

where $\diamond$ denote the Riesz Diamond differential operator.
Lemma 3. Let \( \alpha \) be a complex number and let \( k \) be a non-negative integer. Then

\[
\mathfrak{F}[H_{\alpha} * H_{-4k}] = \mathfrak{F}[H_{\alpha}] \mathfrak{F}[H_{-4k}]
\]

(II.15)

and

\[
H_{\alpha} * H_{-4k} = H_{\alpha - 4k}
\]

(II.16)

Here \(*\) designates, as usual, the convolution.

Proof. The existence of the convolution product \( H_{\alpha} * H_{-4k} \) follows taking into account that \( H_{-4k} \) is a convolutor on \( \mathcal{D}' \) and in particular \( H_{-4k} \in S' \), the space of the temperate distributions. Therefore \( H_{\alpha} \in S \), for all \( \alpha \in \mathbb{C} \).

Let us consider the right hand side of (II.15)

\[
(2\pi)^{\frac{d}{2}} \mathfrak{F}[H_{\alpha}] \mathfrak{F}[H_{-4k}] = (G \mp i0)^{-\frac{d}{2}} \left( \frac{1}{(2\pi)^{\frac{d}{2}}} (G \mp i0)^{k} \right) = \frac{1}{(2\pi)^{\frac{d}{2}}} (G \mp i0)^{-\frac{(\alpha-4k)}{4}}
\]

(II.17)

In the other hand we have

\[
\mathfrak{F}[H_{\alpha-4k}] = \frac{1}{(2\pi)^{\frac{d}{2}}} (G \mp i0)^{-\frac{(\alpha-4k)}{4}}
\]

(II.18)

From (II.14) and (II.15) we have

\[
\mathfrak{F}[H_{\alpha}] \mathfrak{F}[H_{-4k}] = \mathfrak{F}[H_{\alpha-4k}]
\]

(II.19)

Notice that the Fourier transform maps the convolution product to the pointwise product, we have

\[
H_{\alpha} * H_{-4k} = H_{\alpha - 4k}
\]

(II.20)

and the Lemma follows.

Lemma 4. Let \( \alpha \) and \( \beta \) be complex numbers that are different from \( n + 2k \), \( k = 0, 1, 2, \ldots \).

Then, is valid

\[
H_{\alpha} * H_{\beta} = H_{\alpha+\beta}
\]

(II.21)

Proof. It follows from the previous Lemmas.

From the above Lemmas we can obtain the family of the generalized functions that are elementary solutions of the Riesz Diamond differential operator iterated \( k \)-times.
Theorem 1 The distribution \((-1)^k H_{4k}(G \pm i0, n)\); for \(4k \neq n + 2r\), \(r = 0, 1, 2\); are elementary solution of the homogeneous Riesz Diamond differential operator iterated \(k\) times, i.e.

\[ (-1)^k \nabla^k H_{4k}(G \pm i0, n) = \delta \quad (\text{II.22}) \]

Proof. From (II.15) we have \(H_\alpha \ast H_{-4k} = H_{\alpha - 4k}\), valid for \(\alpha \neq n + 2r\), \(r = 0, 1, 2\); and from (II.9) \(H_{-4k} = (-1)^k \nabla^k \delta\). Then

\[ H_\alpha \ast (-1)^k \nabla^k \delta = H_{\alpha - 4k} \]

\[ (-1)^k \nabla^k (H_\alpha \ast \delta) = H_{\alpha - 4k} \]

\[ (-1)^k \nabla^k H_\alpha = H_{\alpha - 4k}, \text{ or} \]

\[ \nabla^k H_\alpha = (-1)^k H_{\alpha - 4k} \quad (\text{II.23}) \]

Taking \(\alpha = 4k\), it result

\[ (-1)^k \nabla^k H_{4k} = H_0 = \delta \quad (\text{II.24}) \]

that is precisely the thesis of Theorem 1.

II.1 Equivalence between our solution of the Riesz Diamond differential operator and the one due to Kananthai

Kananthai (cf.[4]) proved that the distribution \(u(x) = (-1)^k R_{2k}^e(x) \ast R_{2k}^H(x)\), where \(R_{2k}^e(x)\) is given by (I.6) for \(\alpha = 2k\), and \(R_{2k}^H(x)\) is given by (I.7) for \(\beta = 2k\), is an elementary solution of the equation

\[ \nabla^k u(x) = \delta. \]

We had prove that \(H_{4k}(G \pm i0, n)\) is also an elementary solution de \(\nabla^k u(x) = \delta\).

To prove the equivalence between both solution we will prove the identity of its Fourier transform.

From Kananthai (cf.[5]) we know that
\[ \mathcal{F}[(−1)^k R_{2k}^e(x) * R_{2k}^H(x)] = \frac{1}{(2\pi)^\frac{d}{2}} \frac{1}{(r^2)^{kP^k}} \] 

\text{(II.25)}

By using our notation it result

\[ \mathcal{F}[(−1)^k R_{2k}^e(x) * R_{2k}^H(x)] = \frac{1}{(2\pi)^\frac{d}{2}} \frac{1}{(r^2)^{kP^k}} \] 

\text{(II.26)}

Let us now consider the Fourier transform of \(H_{4k}(G \pm i0)\). From (II.6)

\[ \mathcal{F}[H_{4k}(G \pm i0)] = \frac{1}{(2\pi)^\frac{d}{2}} \frac{1}{(r^2)^{kP^k}} \] 

\text{(II.27)}

Then, from (III.2) and (III.3) we have that \((-1)^k R_{2k}^e(x) * R_{2k}^H(x)\) is equivalent to \(H_{4k}(G \pm i0)\), and they are elementary solution of the Riesz diamond operator iterated \(k\) times.

### III Remark

From the definitory formula (II.1) that we many write

\[ \mathcal{P}(m, x) = \left( \sum_{i=1}^{p} x_i^2 \right)^m - \left( \sum_{i=p+1}^{p+q} x_i^2 \right)^m \] 

\text{(III.1)}

for \(m = 1, 2\); we obtain that the generalized function \((G \pm i0)^\lambda\) given by (II.3) when \(m = 1\) reduce to the well known \((P \pm i0)^\lambda\) distribution given by (I.12).

We re write the Fourier transform of \((\mathcal{P}(m, x) \pm i0)^\lambda\),

\[ \mathcal{F}[(\mathcal{P}(m, x) \pm i0)^\lambda] = C(\lambda, m, n)(G(m, x) \mp i0)^{-\lambda - \frac{m}{2m}} \] 

\text{(III.2)}

where

\[ G(m, x) = \left( \sum_{i=1}^{p} \xi_i^2 \right)^m - \left( \sum_{i=p+1}^{p+q} \xi_i^2 \right)^2 \] 

\text{(III.3)}
and

\[ C(\lambda, m, n) = \frac{e^{\pm \frac{i}{2} q} \ 2^{2m\lambda+n} \ \pi^{\frac{n}{2}} \ \Gamma(m\lambda + \frac{n}{2})}{\Gamma(-m\lambda)} \quad (III.4) \]

Let

\[ H_\alpha (\mathcal{P}(m, x) \pm i0, n) = \frac{e^{\mp \frac{i}{2} q} \ \Gamma\left(\frac{n-\alpha}{2}\right)}{2^{\alpha+\frac{1}{2}} \ \pi^{\frac{n}{2}} \ \Gamma\left(\frac{\alpha}{2}\right)} \ (\mathcal{P}(m, x) \pm i0)^{\frac{\alpha-n}{2m}} \quad (III.5) \]

We may observe particular cases of (IV.5).

First: If \( m = 1 \), and the number of negative terms of \( \mathcal{P}(m, x) \), \( q = 0 \), we obtain the elliptic kernel of Marcel Riesz given by (I.6).

Second: If \( m = 1 \), \( q \neq 0 \), we obtain the causal Riesz kernel given by (I.16).

Third: When \( m = 2 \), \( q = 0 \) we obtain a kernel that is an elementary solution of the iterated Laplacian.

Forth: When \( m = 2 \) and \( q \neq 0 \) we obtain the causal Riesz Diamond kernel.

Fifth: When \( \alpha = 2k \), \( m = 1 \) and \( q \neq 0 \), we have \( H_{2k}(P \pm i0, n) \) that are elementary solution of the ultrahyperbolic differential operator.

Sixth: When \( \alpha = 4k \), \( m = 2 \) and \( q \neq 0 \), we have \( H_{4k}(G \pm i0, n) \) that are elementary solution of the Riesz Diamond differential operator.

**References**


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