Pre A*-Algebra with Order Relation

A. Satyanarayana\textsuperscript{1} and J. Venkateswara Rao\textsuperscript{2}

\textsuperscript{1} ANR College, Gudiwada, A.P.
\textsuperscript{2} Mekelle University, Mekelle, Ethiopia
asnmat1969@yahoo.in

Abstract: In this paper we define a partial order \( \leq_c \) on Pre A*-algebra \( A \). Some important properties of \((A, \leq_c)\) are derived which leads to number of equivalent conditions \( A \) to become a Boolean algebra in terms of this partial ordering.

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INTRODUCTION

In 1948 the study of lattice theory had been made by Birkhoff [7]. In a draft paper [3], The Equational Theory of Disjoint Alternatives around 1989, E.G.Maines introduced the concept of Ada(Algebra of disjoint alternatives) \((A, \wedge, \vee, \overline{\cdot}, (\cdot), 0, 1, 2)\) which is however differ from the definition of the Ada of his later paper [4] Adas and the equational theory of if-then-else in 1993. While the Ada of the earlier draft seems to be based on extending the If-Then –Else concept more on the basis of Boolean Algebra and the later concept is based on C-algebra \((A, \wedge, \vee, \cdot)\) introduced by Fernando Guzman and Craig.C. Squir[1].

In 1994, P. Koteswara Rao[2] first introduced the concept A*-Algebra \((A, \wedge, \vee, *, (\cdot), (\cdot), 0, 1, 2)\) not only studied the equivalence with Ada, C-algebra, Ada’s connection with 3- Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. In 2000, J. Venkateswara Rao [5] introduced the concept of Pre A*-algebra \((A, \wedge, \vee, (\cdot))\) as the variety generated by the 3-element algebra \( A = \{0, 1, 2\} \) which is an algebraic form of three valued
conditional logic. It was proved that the only sub directly irreducible Pre A*-algebra are either A or two element Boolean algebra B={0,1}. In [8] Rao et al. generated Pre A*-algebras from Boolean algebras and defined congruence relation and Ternary operation on A. In [6], Venkateswara Rao and Srinivasa Rao defined a partial ordering on a Pre A*-algebra A and the properties of A as a poset are studied.

In this paper we define a partial order ≤ on Pre A*-algebra A. Some important properties of (A, ≤) are derived which leads to number of equivalent conditions A to become a Boolean algebra in terms of this partial ordering.

§ 1. PRELIMINARIES:

1.1 Definition: Boolean algebra is an algebra \( (B, \vee, \wedge, (-)' , 0, 1) \) with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies:
   (i) \( (B, \vee, \wedge) \) is a distributive lattice
   (ii) \( x \wedge 0 = 0, x \vee 1 = 1 \)
   (iii) \( x \wedge x' = 0, x \vee x' = 1 \)

   We can prove that \( x'' = x \), \( (x \vee y)' = x' \wedge y' \), \( (x \wedge y)' = x' \vee y' \) for all \( x, y \in B \).

1.2. Definition: An algebra \( (A, \wedge, \vee, (-)') \) satisfying
   (a) \( x'' = x \), \( \forall x \in A \),
   (b) \( x \wedge x = x \), \( \forall x \in A \),
   (c) \( x \wedge y = y \wedge x \), \( \forall x, y \in A \),
   (d) \( (x \wedge y)' = x' \vee y' \), \( \forall x, y \in A \),
   (e) \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \), \( \forall x, y, z \in A \),
   (f) \( x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \), \( \forall x, y, z \in A \),
   is called a Pre A*-algebra.

1.3. Example: \( 3 = \{0, 1, 2\} \) with operations \( \wedge, \vee, (-)' \) defined below is a Pre A*-algebra.

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<tr>
<th>( \wedge )</th>
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1.4. **Note**: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) \(2^\sim = 2\)  
(b) \(1 \land x = x\) for all \(x \in 3\)  
(c) \(0 \lor x = x\) for all \(x \in 3\)  
(d) \(2 \land x = 2 \lor x = 2\) for all \(x \in 3\).

1.5. **Example**: \(2 = \{0, 1\}\) with operations \(\land, \lor, (-)\) defined below is a Pre A*-algebra.

- \(\land\) table:
  - 0 0 0
  - 0 1 1
  - 1 1 1

- \(\lor\) table:
  - 0 0 0
  - 1 1 1
  - 1 1 1

1.6. **Note**:

(i) \(2, \land, \lor, (-)\) is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The identities 1.1(a) and 1.1(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.1(b) to 1.1(g).

1.7. **Note**: Let \(A\) be a Pre A*-algebra then \(A\) is Boolean algebra iff \(x \lor (x \land y) = x\), \(x \land (x \lor y) = x\) (absorption laws holds)

1.8. **Lemma**: Every Pre A*-algebra satisfies the following laws.

(a) \(x \lor (x^\sim \land x) = x\)  
(b) \((x \lor x^\sim) \land y = (x \land y) \lor (x^\sim \land y)\)  
(c) \((x \lor x^\sim) \land x = x\)  
(d) \((x \lor y) \land z = (x \land z) \lor (x^\sim \land y \land z)\)

1.9. **Definition**: Let \(A\) be a Pre A*-algebra. An element \(x \in A\) is called central element of \(A\) if \(x \lor x^\sim = 1\) and the set \(\{x \in A : x \lor x^\sim = 1\}\) of all central elements of \(A\) is called the centre of \(A\) and it is denoted by \(B(A)\). Note that if \(A\) is a Pre A*-algebra with \(1\) then \(1, 0 \in B(A)\). If the centre of Pre A*-algebra coincides with \(\{0, 1\}\) then we say that \(A\) has trivial centre.

1.10. **Theorem**: Let \(A\) be a Pre A*-algebra with \(1\), then \(B(A)\) is a Boolean algebra with the induced operations \(\land, \lor, (-)\) defined below.

1.11. **Lemma**: Let \(A\) be a Pre A*-algebra with \(1\),

(a) If \(y \in B(A)\) then \(x \land x^\sim \land y = x \land x^\sim, \forall x \in A\)  
(b) \(x \land (x \lor y) = x \lor (x \land y) = x\) if and only if \(x, y \in B(A)\)
2. Partial ordering on Pre A*-Algebra

2.1 Definition: Let A be a Pre A*-algebra define a relation $\leq_c$ on A by $x \leq_c y$ iff $x \wedge y = x$ and $x \vee y = y$

2.2 Lemma: Let A be a Pre A*-algebra then $(A, \leq_c)$ is a poset

Proof: Since $x \vee x = x \wedge x = x, x \leq_c x$, for all $x \in A$

Therefore $\leq_c$ is reflexive.

Suppose $x \leq_c y, y \leq_c z$, for all $x, y, z \in A$

Then $x \wedge y = x$ and $x \vee y = y$; $y \wedge z = y$ and $y \vee z = z$

Now $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$

$x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$

That is $x \leq_c z$, this shows that $\leq_c$ is Transitive.

Let $x \leq_c y$ and $y \leq_c x$ for all $x, y \in A$

Then $x \wedge y = x$ and $x \vee y = y$; $y \wedge x = y$ and $y \vee x = x$

$\therefore x = y$

This shows that $\leq_c$ is anti symmetric. Therefore $(A, \leq_c)$ is a poset.

2.3 Note: Let A be a Pre A*-algebra with 0, 1, 2 then $0 \leq_c 1 \quad (0 \wedge 1 = 0, 0 \vee 1 = 1)$

The Hasse diagram of the poset $(A, \leq_c)$ is

2.4 Note: We have $A \times A = \{ a_1 = (1,1), \ a_2 = (1,0), \ a_3 = (1,2), \ a_4 = (0,1), \ a_5 = (0,0), \ a_6 = (0,2), \ a_7 = (2,1), \ a_8 = (2,0), \ a_9 = (2,2) \}$ is a Pre A*-algebra under point wise operation. The Hasse diagram is of the poset $(A \times A, \leq_c)$ given below
2.5 Note: If A be a Pre \(A^*-\)algebra then the supremum and infimum of any two elements \(x, y\) in the partially ordered set \((A, \leq_c)\) if it exist will be denoted by \(\sup_c\{x, y\}\) and \(\inf_c\{x, y\}\) respectively.

2.6 Theorem: Let A be a Pre \(A^*-\)algebra for any \(x \in A\) then the following holds in \((A, \leq_c)\)

(i) \(x \vee x^-\) is the supremum of \(\{x, x^-\}\)

(ii) \(x \wedge x^-\) is the infimum of \(\{x, x^-\}\)

Proof:

(i) \(x \wedge (x \vee x^-) = x\) and \(x \vee (x \vee x^-) = x \vee x^-\)

Therefore \(x \leq_c x \vee x^-\)

\(x^- \wedge (x \vee x^-) = x^-\) and \(x^- \vee (x \vee x^-) = x \vee x^-\)

Therefore \(x^- \leq_c x \vee x^-\)

\(x \vee x^-\) is upper bound of \(\{x, x^-\}\)

Let \(k\) be the upper bound of \(\{x, x^-\}\)

\(\Rightarrow x \leq_c k\) and \(x^- \leq_c k\) that is \(x \wedge k = x\) and \(x \vee k = k\); \(x^- \wedge k = x^-\) and \(x^- \vee k = k\)

Now \(k \wedge (x \vee x^-) = (k \wedge x) \vee (k \wedge x^-) = x \vee x^-\)

\(k \vee (x \wedge x^-) = (k \vee x) \wedge x^- = k \vee x^- = k\)

\(\therefore x \vee x^- \leq_c k\)

Therefore \(x \vee x^-\) is least upper bound of \(\{x, x^-\}\)

\(\sup_c \{x, x^-\} = x \vee x^-\)

(ii) \(x \wedge (x \wedge x^-) = x \wedge x^-\) and \(x \vee (x \wedge x^-) = x\)

Therefore \(x \wedge x^- \leq_c x\)

\(x^- \wedge (x \wedge x^-) = x \wedge x^-\) and \(x^- \vee (x \wedge x^-) = x^-\)
Therefore \( x \land x' \leq x^- \)

\( x \land x^- \) is lower bound of \( \{x, x^-\} \)

Let \( l \) be the lower bound of \( \{x, x^-\} \)

\[ \Rightarrow l \leq x \text{ and } l \leq x^- \]

Now \( l \land (x \land x^-) = (l \land x) \land x^- = l \land x^- \)

\[ l \lor (x \land x^-) = (l \lor x) \lor (l \land x^-) = x \land x^- \]

\[ \therefore l \leq x \land x^- \]

\[ \therefore x \land x^- \text{ is greatest lower bound of } \{x, x^-\} \]

\( \inf_c \{x, x^-\} = x \land x^- \)

2.7 Lemma: In the poset \((A, \leq_c)\) and \(x, y \in A\). If \(x \leq_c y\) then for \(a \in A\)

(i) \(a \land x \leq_c a \land y\)

(ii) \(a \lor x \leq_c a \lor y\)

Proof: If \(x \leq_c y\) then \(x \land y = x\) and \(x \lor y = y\)

(i) \((a \land x) \land (a \land y) = a \land (x \land y) = a \land x\)

\((a \land x) \lor (a \land y) = a \land (x \lor y) = a \land y\)

\[ \therefore a \land x \leq_c a \land y \]

(ii) \((a \lor x) \land (a \lor y) = a \lor (x \land y) = a \lor x\)

\((a \lor x) \lor (a \lor y) = a \lor (x \lor y) = a \lor y\)

\[ \therefore a \lor x \leq_c a \lor y \]

2.8 Note: 0,2 \(\in A\) but \(0 \lor 2 = 2\) is not upper bound of \(\{0,2\}\) so we have the following theorems

2.9 Lemma: Let \(A\) be a Pre A*-algebra for any \(x, y \in A\), \(x \leq_c x \lor y\) then \(x \lor y\) is the upper bound \(\{x, y\}\)

Proof: Suppose \(x \leq_c x \lor y\) then \(x \lor y\) is upper bound of \(x\).

Since \(x \leq_c x \lor y\) we have \(x \land (x \lor y) = x\) and \(x \lor (x \lor y) = x \lor y\)

Now \((x \lor y) = y\) and also \(y \lor (x \lor y) = x \lor y\)

\[ \therefore y \leq_c x \lor y \]

Therefore \(x \lor y\) is the upper bound of \(x\)

\(x \lor y\) is the upper bound \(\{x, y\}\)

2.10 Lemma: Let \(A\) be a Pre A*-algebra for any \(x, y \in A\), \(x \land y \leq_c y\) then \(x \land y\) is the lower bound of \(\{x, y\}\)
Proof: Suppose $x \wedge y \leq y$ then $x \wedge y$ is lower bound of $y$

Since $x \wedge y \leq y$ we have $(x \wedge y) \wedge y = x \wedge y$ and $(x \wedge y) \vee y = y$.
Now $(x \wedge y) \wedge x = (x \wedge y)$ and $(x \wedge y) \vee x = x \wedge y \leq x$.
Therefore $x \wedge y$ is lower bound of $x$.
\[\therefore x \wedge y \text{ is the lower bound of } \{x, y\}\]

2.11 Theorem: Let $A$ be a Pre $A^*$-algebra for any $x, y \in B(A)$ in the poset $(A, \leq)$ then

(i) $\sup_c \{x, y\} = x \vee y$
(ii) $\inf_c \{x, y\} = x \wedge y$

Proof:

(i) Since $x, y \in B(A)$ then by lemma 1.11(b) we have $x \wedge (x \vee y) = x$ and $x \vee (x \vee y) = x \vee y$.
\[\therefore x \leq x \vee y\]
Similarly $y \wedge (x \vee y) = x$ and $y \vee (x \vee y) = x \vee y$.
\[\therefore y \leq x \vee y\]
Therefore $x \vee y$ is the upper bound of $\{x, y\}$.

Let $k$ be the upper bound of $\{x, y\}$
\[\Rightarrow x \leq k \text{ and } y \leq k \text{ that is } x \wedge k = x \text{ and } x \vee k = k; \quad y \wedge k = y \text{ and } y \vee k = k\]

Now $k \wedge (x \vee y) = (k \wedge x) \vee (k \wedge y) = x \vee y$
\[k \vee (x \vee y) = (k \vee x) \vee y = k \vee y = k\]
\[\therefore x \vee y \leq k\]
Therefore $x \vee y$ is least upper bound of $\{x, y\}$.
\[\sup_c \{x, y\} = x \vee y\]

(ii) Since $x, y \in B(A)$ then by lemma 1.11(b) we have $x \wedge (x \wedge y) = x \wedge y$ and $x \vee (x \wedge y) = x$.
\[\therefore x \wedge y \leq x\]
Similarly $y \wedge (x \wedge y) = x \wedge y$ and $y \vee (x \wedge y) = y$.
\[\therefore x \wedge y \leq y\]
\[\therefore x \wedge y \text{ is the lower bound of } \{x, y\}\]

Let $l$ be the lower bound of $\{x, y\}$
\[\Rightarrow l \leq x \text{ and } l \leq y \text{ that is } x \wedge l = l \text{ and } x \vee l = x; \quad y \wedge l = l \text{ and } y \vee l = y\]

Now $l \wedge (x \wedge y) = (l \wedge x) \wedge y = l \wedge y = l$
\[l \vee (x \wedge y) = (l \vee x) \wedge y = x \wedge y\]
\[\therefore l \leq x \wedge y\]
\[\therefore x \wedge y \text{ is greatest lower bound of } \{x, y\}\]
\[\inf_c \{x, y\} = x \wedge y\]
2.12 Theorem: If $A$ is a Pre A*-algebra and for any $x, y \in B(A)$ then $(A, \leq_c)$ is a lattice.

Proof: By above theorem every pair of elements have infimum and supremum. Hence $(A, \leq_c)$ is a lattice.

2.13 Lemma: If $A$ is a Pre A*-algebra and for any $x, y \in B(A)$ then:

(i) $x \lor y \leq_c x \lor x^c$  
(ii) $x \land x^c \leq_c x \land y$  

Proof: Since $x, y \in B(A)$ then by lemma 1.11(a) we have $x \land x^c \land y = x \land x^c$

(i) Consider $(x \lor x^c) \land (x \lor y) = x \lor (x^c \land y) = x \lor y$

$(x \lor x^c) \lor (x \lor y) = x \lor (x^c \lor y) = x \lor x^c$

\[\therefore x \lor y \leq_c x \lor x^c\]

(ii) Consider $(x \land x^c) \land (x \land y) = x \land (x^c \land y) = x \land x^c$

$(x \land x^c) \lor (x \land y) = x \land (x^c \lor y) = x \land y$

\[\therefore x \land x^c \leq_c x \land y\]

Now we present a number of equivalent conditions for a Pre A*-algebra become a Boolean algebra.

2.14 Theorem: The following conditions are equivalent for any Pre A*-algebra $(A, \land, \lor, (-^c))$:

(1) $A$ is Boolean algebra
(2) $x \leq_c x \lor y$ for all $x, y \in A$
(3) $y \leq_c x \lor y$ for all $x, y \in A$
(4) $x \lor y$ is an upper bound of $\{x, y\}$ in $(A, \leq_c)$ for all $x, y \in A$
(5) $x \lor y$ is an supremum of $\{x, y\}$ in $(A, \leq_c)$ for all $x, y \in A$
(6) $x \lor x^c$ is the greatest element in $(A, \leq_c)$ for every $x \in A$

Proof: (1) $\Rightarrow$ (2) Suppose $A$ be a Boolean algebra

Now $x \land (x \lor y) = x$ and $x \lor (x \lor y) = x \lor y$ (by absorption law)

\[\therefore x \leq_c x \lor y\]

(2) $\Rightarrow$ (3) Suppose $x \leq_c x \lor y$ then $x \land (x \lor y) = x$ and $x \lor (x \lor y) = x \lor y$

Now $y \land (x \lor y) = y$ and $y \lor (x \lor y) = x \lor y$ Therefore $y \leq_c x \lor y$

(3) $\Rightarrow$ (4) Suppose that $y \leq_c x \lor y \Rightarrow y \land (x \lor y) = y$ and $y \lor (x \lor y) = x \lor y$

Since $y \leq_c x \lor y$ then $x \lor y$ is upper bound of $y$

Now $x \land (x \lor y) = x$ (by supposition) and $x \lor (x \lor y) = x \lor y$

$x \leq_c x \lor y \Rightarrow x \lor y$ is upper bound of $x$
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\[ x \lor y \] is an upper bound of \( \{x, y\} \)

(4) \( \Rightarrow \) (5) Suppose \( x \lor y \) is an upper bound of \( \{x, y\} \)

Suppose \( z \) is an upper bound of \( \{x, y\} \), then \( x \leq_z z \), \( y \leq_z z \)

\[ x \land z = x \quad \text{and} \quad x \lor z = z \quad y \land z = y \quad \text{and} \quad y \lor z = z \]

Now \( z \land (x \lor y) = (z \land x) \lor (z \land y) = x \lor y \)

Therefore \( x \lor y \leq_z z \).

\( x \lor y \) is least upper bound of \( \{x, y\} \)

Hence \( \sup_{ \leq } \{x, y\} = x \lor y \)

(5) \( \Rightarrow \) (6) Suppose \( \sup_{ \leq } \{x, y\} = x \lor y \) then \( x, y \in B(A) \)

Now \( \sup_{ \leq } \{ x \lor x^-, y \} = x \lor x^- \lor y = x \lor x^- \) (by lemma 1.10(a))

\[ y \leq x \lor x^- \therefore x \lor x^- \] is the greatest element in \( (A, \leq) \)

2.15 Theorem: The following conditions are equivalent for any Pre A*-algebra \( (A, \land, \lor, (\cdot^\ast)) \)

(1) \( A \) is Boolean algebra
(2) \( x \land y \leq_x x \) for all \( x, y \in A \)
(3) \( x \land y \leq_x y \) for all \( x, y \in A \)
(4) \( x \land y \) is a lower bound of \( \{x, y\} \) in \( (A, \leq_x) \) for all \( x, y \in A \)
(5) \( x \land y \) is an infimum of \( \{x, y\} \) in \( (A, \leq_x) \) for all \( x, y \in A \)
(6) \( x \lor x^- \) is the least element in \( (A, \leq_x) \) for every \( x \in A \)

Proof: (1) \( \Rightarrow \) (2) Suppose \( A \) be a Boolean algebra

Now \( x \land (x \lor y) = x \land y \) and \( x \lor (x \land y) = x \) (by absorption law)

\[ \therefore x \land y \leq_x x \]

(2) \( \Rightarrow \) (3) Suppose \( x \land y \leq_x x \) then \( x \land (x \lor y) = x \land y \) therefore \( x \lor (x \land y) = x \)
Now \( y \land (x \land y) = x \land y \) and \( y \lor (x \land y) = y \). Therefore \( x \land y \leq y \)

\((3) \Rightarrow (4)\) Suppose that \( x \land y \leq y \) \( \Rightarrow \) \( y \land (x \land y) = x \land y \) and \( y \lor (x \land y) = y \)

Since \( x \land y \leq y \) then \( x \land y \) is lower bound of \( y \)

Now \( x \land (x \land y) = x \land y \) and \( x \lor (x \land y) = x \) (by supposition)

\( \therefore x \land y \leq x \)

\[ \Rightarrow x \land y \text{ is a lower bound of } x \]

\( x \land y \) is a lower bound of \( \{x, y\} \)

\((4) \Rightarrow (5)\) Suppose \( x \land y \) is a lower bound of \( \{x, y\} \)

Suppose \( z \) is a lower bound of \( \{x, y\} \) then \( z \leq x, z \leq y \)

\[ \Rightarrow x \land z = z \text{ and } x \lor z = x; y \land z = z \text{ and } y \lor z = y \]

Now \( z \land (x \land y) = (z \land x) \land y = z \land y = z \)

\( z \lor (x \land y) = (z \lor x) \land (z \lor y) = x \lor y \)

Therefore \( z \leq x \land y \)

\( x \land y \) is the greatest lower bound of \( \{x, y\} \)

Hence \( \inf_c \{x, y\} = x \land y \)

\((5) \Rightarrow (6)\) Suppose \( \inf_c \{x, y\} = x \land y \) then \( x, y \in B(A) \)

Now \( \inf_c \{x \land x^-, y\} = x \land x^- \land y = x \land x^- \) (by lemma 1.10(a))

\[ \Rightarrow x \land x^- \leq y \text{ therefore } x \land x^- \text{ is the least element in } (A, \leq_c) \]

\((6) \Rightarrow (1)\) Suppose \( x \land x^- \) is the least element in \( A \) then \( x \land x^- \leq y \), for \( y \in A \)

\[ \Rightarrow (x \land x^-) \land y = x \land x^- \text{ and } (x \land x^-) \lor y = y \]

Now \( y \land (x \lor y) = [(x \land x^-) \lor y] \land (x \lor y) = [ (x \land x^-) \land x ] \lor y = (x \land x^-) \lor y = y \)

( by supposition)

Therefore by Lemma 1.7 we have \( B \) is Boolean algebra

References

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