Kernels by Monochromatic Directed Paths in 3-Colored Tournaments and Quasi-Tournaments

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Abstract

We call the tournament T an m-coloured tournament if the arcs of T are coloured with m colours. If v is a vertex of an m-coloured tournament T, we denote by $\zeta(v)$ the set of colours assigned to the arcs with v as endpoint, $\zeta^{-}(v)$ the set of colours assigned to the arcs with v as final vertex, and $\zeta^{+}(v)$ the set of colours assigned to the arcs with v as an initial vertex.

In this paper, we prove that if T is a 3-coloured tournament which does not contain $C_3$ such that:

1. for each $v \in V(T) \mid \zeta^{-}(v) \mid \leq 2$,
2. $\zeta^{-}(w) = \{a, b\}$ for some $w \in V(T)$, with $a \neq b$, implies that $\zeta^{+}(w) = \{c\}$, where $c \notin \{a, b\}$,

then T has a kernel by monochromatic paths.

We prove that if D is an m-coloured digraph, $m \geq 4$, resulting from the deletion of the single arc $(x, y)$ from some m-coloured tournament such that $|\zeta(v)| \leq 2$ for each $v \in V(D)$, then D has a kernel by monochromatic paths.

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1 Introduction

We refer the reader to [1] for general concepts. For a digraph D, the vertex set is denoted by $V(D)$ and the arc set by $F(D)$. A subdigraph $D_1$ of D is an...
spanning subdigraph if \( V(D_1) = V(D) \). If \( S \) is a nonempty subset of \( V(D) \), then the subdigraph of \( D \) induced by the vertex set \( S \), \( D[S] \), is that digraph having vertex set \( S \), whose arc set consist of all those arcs of \( D \) joining vertices of \( S \). An arc \((u_1, u_2) \in F(D)\) is called an asymmetrical arc (symmetrical arc) if \((u_2, u_1) \notin F(D)\) \(((u_2, u_1) \in F(D))\). The asymmetrical part of \( D \) (symmetrical part of \( D \)) denoted by \( Asym(D) \) (\( Sym(D) \)) is the spanning subdigraph of \( D \) whose arcs are the asymmetrical arcs (symmetrical arcs) of \( D \). \( D \) is called an asymmetrical digraph if \( Asym(D) = D \). \( D \) is called complete if for every two distinct vertices \( u \) and \( v \) of \( D \), at least one of the arcs \((u, v)\) or \((v, u)\) is present in \( D \). A tournament is a complete asymmetrical digraph. The arc \((u_1, u_2) \in F(D)\) is called an \( S_1S_2 \)-arc whenever \( u_1 \in S_1 \subseteq V(D) \) and \( u_2 \in S_2 \subseteq V(D) \).

A set \( I \subseteq V(D) \) is an independent set in \( D \) if \( F(D[I]) = \emptyset \). If \( W \) is a directed path or directed cycle in \( D \), then \( l(W) \) will denote its length. For \( \{u_1, u_2\} \subseteq V(D) \) an \( u_1u_2 \)-walk is a directed walk from \( u_1 \) to \( u_2 \) in \( D \) and if we restrict \( u_1 \) and \( u_2 \) to \( V(W) \), then the \( u_1u_2 \)-walk contained in \( W \) will be denoted by \((u_1, W, u_2)\). If \( S \subseteq V(D) \) and \( u \in V(D) \), then an \( uS \)-walk is an \( ux \)-walk for some \( x \in S \).

A kernel \( N \) of \( D \) is an independent set of vertices such that for each \( z \in V(D) \setminus N \) there exists a \( zN \)-arc in \( D \). A digraph \( D \) such that every induced subdigraph in \( D \) has a kernel is called a kernel-perfect digraph (or \( KP \)-digraph).

We call the digraph \( D \) an \( m \)-coloured digraph if the arcs of \( D \) are coloured with \( m \) colours. Let \( T_3 \) and \( C_3 \), respectively, denote the transitive tournament of order 3 and the 3-cycle, both of whose arcs are coloured with three distinct colours. Let \( v \in V(D) \), we denote by \( \zeta(v) \) the set of colours assigned to the arcs with \( v \) as endpoint, \( \zeta^-(v) \) the set of colours assigned to the arcs with \( v \) as final vertex, and \( \zeta^+(v) \) the set of colours assigned to the arcs with \( v \) as an initial vertex. A directed path (or a directed cycle) in \( D \) is called monochromatic if all of its arcs are coloured alike. A directed cycle is called a quasi-monochromatic cycle if with at most one exception all of its arcs are coloured alike. Let \( D \) be an \( m \)-coloured digraph and \( \gamma_n = (0, 1, \ldots, n-1, 0) \) a directed cycle of \( D \), we will say that \( \gamma_n \) is \( C(D) \)-monochromatic if there exists a set \( \{f_i = (i, i+1) \in F(C(D)) \mid i \in \{1, \ldots, n\} \} \) notation mod \( n \) of arcs coloured alike.

A set \( N \subseteq V(D) \) is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices \( u, v \in N \), there is no monochromatic directed path between them in \( D \), and (ii) for every vertex \( x \in V(D) \setminus N \), there is a vertex \( y \in N \) such that there is an \( xy \)-monochromatic directed path in \( D \).

For an \( m \)-coloured digraph \( D \), the closure of \( D \), denoted by \( C(D) \), is the digraph defined as follows:
\[ V(\mathcal{C}(D)) = V(D), \]
\[ F(\mathcal{C}(D)) = F(D) \cup \bigcup_{i=1}^{m} \{(u, v) \mid \text{there exist a } uv\text{-monochromatic directed path of colour } i \text{ contained in } D \}. \]

Notice that for any \( m \)-coloured digraph \( D \); \( \mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D) \), and \( D \) has a kernel by monochromatic paths if and only if \( \mathcal{C}(D) \) has a kernel.

In [9], Sands, Sauer and Woodrow have proved that for any 2-coloured digraph \( D \) has a kernel by monochromatic directed paths; in particular they proved that every 2-coloured tournament \( T \) has a vertex \( v \) such that for any other vertex \( x \) of \( T \) there is a monochromatic directed path from \( x \) to \( v \). They also raised the following problem: Let \( T \) be a 3-coloured tournament which does not contain \( C_3 \): Must \( T \) contain a vertex \( v \) such that for every other vertex \( x \) of \( T \) there is a monochromatic directed path from \( x \) to \( v \)? In [8], Shen Minggang proved that if \( T \) is an \( m \)-coloured tournament which does not contain \( C_3 \) or \( T_3 \), then there is a vertex \( v \) of \( T \) such that for every other vertex \( x \) of \( T \) there is a monochromatic directed path from \( x \) to \( v \). He proved that the situation is best possible for \( m \geq 5 \). In [6], H. Galeana-Sánchez and R. Rojas-Monroy constructed a family of counterexamples to the question for \( m = 4 \). They proved in [7] that if in the problem we ask that for every vertex \( v \) of \( T \), \( |\zeta(v)| \leq 2 \); the answer will be yes. In [3] H. Galeana-Sánchez proved that if \( T \) is an \( m \)-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic, then \( \mathcal{C}(T) \) is kernel-perfect and hence \( T \) has a kernel by monochromatic paths.

In this paper, we prove that if \( T \) is a 3-coloured tournament which does not contain \( C_3 \) such that:

1. for each \( v \in V(T) \) \( |\zeta^-(v)| \leq 2 \),
2. \( \zeta^-(w) = \{a, b\} \) for some \( w \in V(T) \), with \( a \neq b \), implies that \( \zeta^+(w) = \{c\} \), where \( c \notin \{a, b\} \),

then \( \mathcal{C}(T) \) is a kernel-perfect digraph and hence \( T \) has a kernel by monochromatic paths. We prove that: the conditions of the Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 (the Theorems are listed below for the readers) do not imply the conditions 1 and 2 of above and vice versa.

Finally under similar conditions as in [7] we prove that: if \( D \) is an \( m \)-coloured digraph, \( m \geq 4 \), resulting from the deletion of the single arc \( (x, y) \) from some \( m \)-coloured tournament such that \( |\zeta(v)| \leq 2 \) for each \( v \in V(D) \), then \( \mathcal{C}(D) \) is kernel-perfect and hence \( D \) has a kernel by monochromatic paths.
For $m = 3$ we construct a counterexample to the question done for Sands, Sauer and Woodrow on quasi-tournaments.

We will need the following results.

**Theorem 1.1.** (Shen Minggang [8])
If $T$ is an $m$-coloured tournament without $C_3$ or $T_3$, then there is a vertex $v \in V(T)$ such that for every other vertex $x$ of $T$ there is a monochromatic directed path from $x$ to $v$.

**Theorem 1.2.** (H. Galeana-Sánchez [3])
Let $T$ be an $m$-coloured tournament. If each directed cycle contained in $T$ and of length at most 4 is a quasi-monochromatic cycle, then $\mathcal{C}(T)$ is a kernel-perfect digraph.

**Theorem 1.3.** (H. Galeana-Sánchez [3])
Let $T$ be an $m$-coloured tournament. If each directed cycle of length 3 contained in $T$ is $\mathcal{C}(T)$-monochromatic, then $\mathcal{C}(T)$ is a kernel-perfect digraph.

**Theorem 1.4.** (H. Galeana-Sánchez [3])
Let $T$ be an $m$-coloured tournament with $p$ vertices. If there exist some $k$ ($3 \leq k \leq p$) such that:

(i) For each $m$-coloured tournament $T' \subseteq T$ such that $T'$ contains no directed cycle of length $k$, $\mathcal{C}(T')$ is a kernel-perfect digraph

(ii) Each directed cycle of length $k$ contained in $T$ is $\mathcal{C}(T)$-monochromatic,

then $\mathcal{C}(T)$ is a kernel-perfect digraph.

**Theorem 1.5.** (R. Rojas-Monroy, H. Galeana-Sánchez [7])
Let $T$ be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic, and for each $v \in V(T)$ we have $|\zeta(v)| \leq 2$, then $\mathcal{C}(T)$ is a kernel-perfect digraph.

**Theorem 1.6.** (Berge and Duchet [2])
A complete digraph is a kernel-perfect digraph if and only if every directed cycle has at least one symmetrical arc.
2 Tournaments 3-coloured

Theorem 2.1. Let $T$ be a 3-coloured tournament which does not contain $C_3$ such that:

1. For each $v \in V(T)$ $|\zeta^-(v)| \leq 2$

2. $\zeta^-(w) = \{a, b\}$ for some $w \in V(T)$, with $a \neq b$, implies that $\zeta^+(w) = \{c\}$, where $c \notin \{a, b\}$,

then $\mathcal{C}(T)$ is a kernel-perfect digraph.

Proof. Without loss of generality we will assume that $T$ is 3-coloured with colours 1, 2 and 3. Observe that for each $v \in V(\mathcal{C}(T))$, the cardinality of the set of colours assigned to the arcs of $\mathcal{C}(T)$ with $v$ as final vertex is at most 2 (because $|\zeta^-(v)| \leq 2$); and if $|\zeta^-(w)| = \{a, b\}$, $a \neq b$, for some $w \in V(T)$, then the set of colours assigned to the arcs of $\mathcal{C}(T)$ with $w$ as initial vertex is equal to the set $\{c\}$, where $c \notin \{a, b\}$ (because, in this case $\zeta^+(w) = \{c\}$, where $c \notin \{a, b\}$).

According to Theorem 1.6 it would be sufficient to prove that every directed cycle contained in $\mathcal{C}(T)$ has at least one symmetrical arc.

If $\gamma$ is a directed cycle contained in $\mathcal{C}(T)$ and $\gamma \notin T$, then there exists $(u, v) \in F(\gamma)$ such that $(u, v) \notin F(T)$. So $(v, u) \in F(T) \subseteq F(\mathcal{C}(T))$, which implies that $(u, v) \in F(\gamma) \cap F(\text{Sym}(\mathcal{C}(T)))$.

Assume that $\gamma$ is a directed cycle in $T$. In this case we proceed by induction on $l(\gamma)$, the length of $\gamma$.

If $l(\gamma) = 3$, then $\gamma = (v_0, v_1, v_2, v_0)$ is a directed cycle of length 3 in $T$ and it is not $C_3$ by the hypothesis of Theorem 2.1. So we would assume without loss of generality that the arcs $(v_0, v_1)$ and $(v_1, v_2)$ are coloured alike. Then $(v_0, v_1, v_2)$ is an monochromatic directed path contained in $T$, which implies that $(v_0, v_2) \in F(\gamma) \cap F(\text{Sym}(\mathcal{C}(T)))$.

Suppose that every directed cycle $\gamma' \subseteq T$ of length at most $n$ has at least one symmetrical arc in $\mathcal{C}(T)$.

Let $\gamma = (v_0, v_1, ..., v_n, v_0)$ be a directed cycle of length $n+1$ contained in $T$, $n \geq 3$.

Suppose, by the contrary, that $\gamma \subseteq \text{Asym}(\mathcal{C}(T))$. 

We first introduce some notation. For distinct vertices \( u, v \) of \( \mathcal{C}(T) \), \( u \rightarrow^a v \) will mean that \((u, v) \in F(T)\) and \((u, v)\) is coloured \( a \). \( u \rightarrow^a v \) will mean that there is an arc from \( u \) to \( v \) coloured \( a \) in \( \mathcal{C}(T) \). The negation of \( u \rightarrow^a v \) will be denoted \( u \rightarrow^a v \). Finally, \( u \Rightarrow^a v \) will mean that \( u \rightarrow^a v \) and all the arcs from \( u \) to \( v \) in \( \mathcal{C}(T) \) are coloured \( a \).

The following statements (over \( \gamma \)) will allow us to get a contradiction.

1. \( \gamma \) is not monochromatic.
   
   This follows from the fact that \( \gamma \subseteq \text{Asym}(\mathcal{C}(T)) \).

2. For every \( i \) and for every \( j \notin \{i - 1, i + 1\} \), it holds that:
   
   \[(v_i, v_j) \in F(\text{Sym}(\mathcal{C}(T))).\]

   Let \( v_i, v_j \in V(\gamma) \), two vertex non consecutive in \( \gamma \). Without loss of generality suppose that \( i < j \). Since \( v_i, v_j \in V(T) \), we have that \( (v_i, v_j) \in F(T) \) or \( (v_j, v_i) \in F(T) \). If \( (v_i, v_j) \in F(T) \), then \( \gamma' = (v_i, v_j) \cup(v_j, v_i, v_i) \) is a directed cycle contained in \( T \) with length at most \( n \), so the inductive hypothesis implies that \( F(\gamma') \cap F(\text{Sym}(\mathcal{C}(T))) \neq \emptyset \). Since \( \gamma \subseteq \text{Asym}(\mathcal{C}(T)) \), we conclude that \( (v_i, v_j) \in F(\text{Sym}(\mathcal{C}(T))) \). If \( (v_j, v_i) \in F(T) \), in the same manner, by considering the directed cycle \((v_j, v_i, v_i, v_j)\), we obtain \( (v_j, v_i) \in F(\text{Sym}(\mathcal{C}(T))) \).

   In what follows the notation will be taken modulo \( n+1 \).

3. Let \( i \in \{0, 1, \ldots, n\} \). If \( v_{i-1} \rightarrow^a v_i \) and \( v_{i-1} \rightarrow^b v_{i+1} \), \( a \neq b \), then \( v_{i+1} \Rightarrow^c v_{i-1} \) and \( (v_{i-1}, v_{i+1}) \in F(T) \), where \( c \notin \{a, b\} \).

   Since \( v_{i-1} \) and \( v_{i+1} \) are not consecutives, it follows from statement (2) that \( (v_{i-1}, v_{i+1}) \in F(\text{Sym}(\mathcal{C}(T))) \). If \( v_{i+1} \rightarrow^a v_{i-1} \), since \( v_{i-1} \rightarrow^a v_i \), then \( v_{i+1} \rightarrow^b v_i \) and so \( (v_i, v_{i+1}) \in F(\gamma) \cap F(\text{Sym}(\mathcal{C}(T))) \), in contradiction with the fact that \( \gamma \subseteq \text{Asym}(\mathcal{C}(T)) \). The case when \( v_{i+1} \rightarrow^b v_{i-1} \) follows in a similar way. Hence, we conclude that \( v_{i+1} \Rightarrow^c v_{i-1} \) for \( c \notin \{a, b\} \). Moreover, it follows from the hypothesis on Theorem 2.1 that the 3-coloured directed cycle \((v_{i-1}, v_i, v_{i+1}, v_{i-1})\) is not contained in \( T \), which implies that \( (v_{i-1}, v_{i+1}) \in F(T) \).

4. If \( a \in \zeta^-(v_i) \cap \zeta^+(v_i) \) for some \( i \in \{0, 1, \ldots, n\} \), then \( \zeta^-(v_i) = \{a\} \).

   If \( |\zeta^-(v_i)| = 2 \), then from the condition 2 of the Theorem 2.1 we have that \( \zeta^+(v_i) = \{c\} \), with \( c \neq a \), contradicting that \( a \in \zeta^+(v_i) \). Hence \( \zeta^-(v_i) = \{a\} \).
5. If \( \{a, b\} \subseteq \zeta^+(v_i) \) and \( c \in \zeta^-(v_i) \) \( (a \neq b, c \not\in \{a, b\}) \) for some \( i \in \{0,1,...,n\} \), then \( \zeta^-(v_i) = \{c\} \).

This follows from the condition 2 on Theorem 2.1.

6. \( v_n \rightarrow^1 v_0 \) and \( v_0 \rightarrow^2 v_1 \).

From the statement (1), there exist two consecutive arcs of \( \gamma \) coloured differently. Without loss of generality let us suppose that \( v_n \rightarrow^1 v_0 \) and \( v_0 \rightarrow^2 v_1 \).

7. \( v_1 \Rightarrow^3 v_n \) and \( (v_n, v_1) \in F(T) \).

This is a consequence of both, the previous statement and the statement (3) applied to \( i = 0 \).

Proposition (7) implies that \( (v_n, v_1) \in F(T) \). We continue the proof by analysing the cases: \( v_n \rightarrow^1 v_1 \), \( v_n \rightarrow^2 v_1 \) or \( v_n \rightarrow^3 v_1 \).

**Case 1.** \( v_n \rightarrow^1 v_1 \).

In this case we have the following assertions:

1.a) \( \zeta^-(v_1) = \{1, 2\} \).

Since \( v_0 \rightarrow^2 v_1 \) (due to (6)) and \( v_n \rightarrow^1 v_1 \), it follows from the condition 1 of the Theorem 2.1 that \( \zeta^-(v_1) = \{1, 2\} \).

1.b) \( v_1 \Rightarrow^3 v_2 \).

This is a consequence of both, the previous statement and the condition 2 of the Theorem 2.1.

1.c) \( v_2 \Rightarrow^1 v_0 \) and \( (v_0, v_2) \in F(T) \).

Since \( v_0 \rightarrow^2 v_1 \) (due to (6)) and \( v_1 \rightarrow^3 v_2 \), by the previous statement. Then, from the statement (3) applied to \( i = 1 \) we have that \( v_2 \Rightarrow^1 v_0 \) and \( (v_0, v_2) \in F(T) \).
1.d) \( n \geq 4 \).

Proceeding by contradiction, suppose that \( n = 3 \).

Notice that \( v_2 \not\rightarrow v_1 \), otherwise \((v_2, v_3, v_1)\) is a monochromatic directed path 1-coloured (see figure 1.A) and so \((v_2, v_1) \in F(\gamma) \cap F(Sym(\mathcal{C}(T)))\), in contradiction with \( \gamma \subseteq Asym(\mathcal{C}(T)) \).

On the other hand, notice that \( v_0 \not\rightarrow v_2 \) and \( v_0 \not\rightarrow v_2 \). If \( v_0 \rightarrow v_2 \), then \( 1 \in \zeta^-(v_2) \cap \zeta^+(v_2) \) (due to (1.c)). It follows from statement (4) that \( \zeta^-(v_2) = \{1\} \). This contradicts that \( 3 \in \zeta^-(v_2) \) (see (1.b)).

Therefore we conclude that \( v_0 \rightarrow v_2 \).

Finally notice that \( v_2 \rightarrow v_3 \); otherwise, since \( v_0 \rightarrow v_2 \), we have that \((v_0, v_2, v_3)\) is a monochromatic directed path 3-coloured, and so \((v_0, v_3) \in F(\gamma) \cap F(Sym(\mathcal{C}(T)))\), in contradiction with the fact \( \gamma \subseteq Asym(\mathcal{C}(T)) \).

Then, due to \( v_2 \rightarrow v_3 \) and \( v_2 \rightarrow v_3 \), we have that \( v_2 \rightarrow v_3 \), and so \((v_0, v_2, v_3, v_0)\) is a \( C_3 \) contained in \( T \). This contradicts the hypothesis of the Theorem 2.1, and so it is proved that \( n \geq 4 \).

1.e) \( v_0 \rightarrow v_2 \).

Suppose, by the contrary, that \( v_0 \rightarrow v_2 \). Then \( 1 \in \zeta^-(v_2) \cap \zeta^+(v_2) \) (due to (1.c)). It follows from statement (4) that \( \zeta^-(v_2) = \{1\} \). This contradicts that \( 3 \in \zeta^-(v_2) \) (see (1.b)).
1.f) \( v_0 \rightarrow v_2 \).

Suppose, by the contrary, that \( v_0 \rightarrow v_2 \). Since \( 3 \in \zeta^-(v_2) \) (see (1.b)), then by the condition 1 of the Theorem 2.1 we have that \( \zeta^-(v_2) = \{2, 3\} \), and so by the condition 2 of the Theorem 2.1 we have \( v_2 \rightarrow v_n \). Since \( v_n \rightarrow v_0 \) (by statement (6)), then \( 1 \in \zeta^-(v_n) \cap \zeta^+(v_n) \). On the other hand, since \( 3 \in \zeta^-(v_n) \) (due to (7)), from the condition 2 of the Theorem 2.1 we get that \( \zeta^-(v_n) = \{1, 3\} \), in contradiction with the statement (4) applied to \( a = 1 \).

1.g) \( v_0 \rightarrow v_2 \).

This is a consequence of both, the previous statement and the statement (1.e).

1.h) \( (v_n, v_2) \in F(T) \).

Suppose, by the contrary, that \( (v_2, v_n) \in F(T) \). Since \( v_n \rightarrow v_0 \) (due to (6)) and \( v_0 \rightarrow v_2 \) (by (1.g)), then \( (v_n, v_0, v_2, v_n) \) is a directed cycle of length 3 contained in \( T \) and it is not \( C_3 \) by the hypothesis of the Theorem 2.1 (see figure 2.A). If \( v_2 \rightarrow v_n \), then we have that \( 1 \in \zeta^-(v_n) \cap \zeta^+(v_n) \), and so \( \zeta^-(v_n) = \{1\} \) (due to (4)) (see figure 2.B), in contradiction with the fact that \( 3 \in \zeta^-(v_n) \) (see (7)). Therefore \( v_2 \rightarrow v_n \), and so \( (v_0, v_2, v_n) \) is a monochromatic directed path of \( \mathcal{C}(T) \) (see figure 2.C) which implies that \( (v_0, v_n) \in F(\gamma) \cap F(Sym(\mathcal{C}(T))) \), in contradiction with \( \gamma \subseteq Asym(\mathcal{C}(T)) \).

1.i) \( v_n \rightarrow v_2 \).

Suppose, by the contrary, that \( v_n \rightarrow v_2 \). Since \( 1 \in \zeta^+(v_2) \) (due to statement (1.e)), then \( 1 \in \zeta^-(v_2) \cap \zeta^+(v_2) \), implying that \( \zeta^-(v_2) = \{1\} \) (by statement (4)). This, however, contradicts the fact that \( 3 \in \zeta^-(v_2) \) (see statement (1.g)).
1.j) \( v_n \to v_2 \).
Suppose, by the contrary, that \( v_n \to v_2 \). Since \( 3 \in \zeta^-(v_2) \) (due to (1.g)), then \( \zeta^-(v_2) = \{2, 3\} \) and so by the condition 2 of the Theorem 2.1 we have that \( v_2 \to v_n \). This implies that \((v_2, v_n, v_1)\) is a monochromatic directed path 1-coloured in \( \mathcal{C}(T) \) (see Case 1). Therefore \((v_2, v_1) \in (F(\gamma) \cap Sym(\mathcal{C}(T)))\), in contradiction to the fact that \( \gamma \subseteq Asym(\mathcal{C}(T)) \).

1.k) \( v_n \Rightarrow 3v_2 \).
This is a consequence of both, the previous statement and the statement (1.i).

Finally we conclude of both, the previous statement and the statement (7) that \( 3 \in \zeta^-(v_n) \cap \zeta^+(v_n) \). So, \( \zeta^-(v_n) = \{3\} \) (due to (4)), implying that \( v_2 \to 3v_n \).
Hence \((v_0, v_2, v_n)\) is a monochromatic directed path 3-coloured which implies that \((v_0, v_n) \in (F(\gamma) \cap Sym(\mathcal{C}(T)))\), in contradiction with \( \gamma \subseteq Asym(\mathcal{C}(T)) \).

Therefore \( v_n \not\Rightarrow 1v_1 \).

**Case 2.** \( v_n \to 2v_1 \).

In this case we have the following assertions:

2.a) \( \zeta^-(v_n) = \{3\} \).
Since \( v_n \to 1v_0 \) (see (6)), it follows that \( \{1, 2\} \subseteq \zeta^+(v_n) \). By (7), \( 3 \in \zeta^-(v_n) \).
Then by statement (5), \( \zeta^-(v_n) = \{3\} \).

2.b) \( n \geq 4 \).
Suppose, by the contrary, that \( n = 3 \). Notice that \((v_3, v_1, v_2, v_3)\) is a directed cycle of length 3 contained in \( T \) and it is not \( C_3 \) by the hypothesis of the Theorem 2.1 (see figure 3.A). Since \( v_2 \to 3v_3 \) (due to (2.a)), then \( v_1 \to v_2 \) or \( v_1 \to 3v_2 \). If \( v_1 \to v_2 \), then \((v_3, v_1, v_2)\) is a monochromatic directed path 2-coloured; so that \((v_3, v_2) \in (F(\gamma) \cap F(Sym(\mathcal{C}(T))))\) (see figure 3.B), contradicting that \( \gamma \subseteq Asym(\mathcal{C}(T)) \). If \( v_1 \to 3v_2 \), then \( 3 \in \zeta^-(v_2) \cap \zeta^+(v_2) \); hence the
statement (4) implies that \( v_0 \rightarrow^3 v_2 \). Consequently, \((v_0, v_2, v_3)\) is a monochromatic directed path 3-coloured (see figure 3.C), which implies that \((v_0, v_3) \in (F(\gamma) \cap F(Sym(\mathcal{C}(T))))\) in contradiction to \( \gamma \subseteq Asym(\mathcal{C}(T)) \).

Figure 3:

2.c) \( v_1 \Rightarrow^2 v_2 \).

By contradiction suppose that \( v_1 \not\Rightarrow^2 v_2 \).

If \( v_1 \rightarrow^1 v_2 \), from (6) and applying statement (3) on \( i = 1 \) we have \( v_2 \rightarrow^3 v_0 \) and \((v_0, v_2) \in F(T)\); so \( \zeta^-(v_0) = \{1, 3\} \) (see (6)). Then by condition 2 of Theorem 2.1, \( v_0 \rightarrow^2 v_2 \) and \( v_0 \rightarrow^2 v_3 \). This, together with the condition 2 of the Theorem 2.1, imply that \( \zeta^-(v_2) = \{1, 2\} \) and \( v_2 \rightarrow^3 v_3 \). Then \( \zeta^-(v_3) = \{2, 3\} \) (see figure 4.A). Therefore \( v_3 \rightarrow^1 v_1 \), contradicting (3) applied on \( i = 2 \).

If \( v_1 \rightarrow^3 v_2 \), since \( v_0 \rightarrow^2 v_1 \), then by (3) \( 1 \in \zeta^+(v_2) \) and \((v_0, v_2) \in F(T)\). Observe that \( v_0 \rightarrow^1 v_2 \), otherwise \( \zeta^-(v_2) = \{1, 2\} \), contradicting that \( 1 \in \zeta^+(v_2) \).

Hence \( v_0 \rightarrow^2 v_2 \) or \( v_0 \rightarrow^3 v_2 \). If \( v_0 \rightarrow^2 v_2 \), then \( \zeta^-(v_2) = \{2, 3\} \), which implies that \( v_2 \rightarrow^1 v_n \) and so \( 1 \in \zeta^-(v_n) \) (see figure 4.B). This, however, contradicts (2.a). If \( v_0 \rightarrow^3 v_2 \), since \((v_2, v_n)\) has colour 3 (by (2.a)), then \((v_0, v_2, v_n)\) is a monochromatic directed path 3-coloured (see figure 4.C), which implies that \((v_0, v_n) \in F(\gamma \cap Sym(\mathcal{C}(T)))\), in contradiction with the fact that \( \gamma \subseteq Asym(\mathcal{C}(T)) \). Thus, \( v_1 \Rightarrow^2 v_2 \).

Let \( j_0 = \min\{j \in \{2, \ldots, n\} \mid v_j \rightarrow^2 v_{j+1}\} \).

\( j_0 \) is well defined because \( v_n \rightarrow^2 v_0 \).

2.d) \( v_i \rightarrow^2 v_{i+1} \) for each \( 0 \leq i \leq j_0 - 1 \).

This follows from the definition of \( j_0 \) and the fact that both \((v_0, v_1)\) and \((v_1, v_2)\) have colour 2.
2.e) \( \zeta^-(v_i) = \{2\} \) for each \( i \in \{1, \ldots, j_0-1\} \).

From (2.d) we have that \( 2 \in \zeta^-(v_i) \cap \zeta^+(v_i) \) for each \( 1 \leq i \leq j_0-1 \). Therefore by statement (4), \( \zeta^-(v_i) = \{2\} \) for each \( i \in \{1, \ldots, j_0-1\} \).

Finally, since \( v_{j_0-1} \to^a v_{j_0} \) (with \( a \neq 2 \)) and \( v_{j_0} \to^2 v_{j_0} \) (by (2.d)), we have by the statement (3) that \( v_{j_0} \to^b v_{j_0-1} \), with \( b \notin \{2, a\} \). Hence \( \zeta^-(v_{j_0-1}) = \{2, b\} \). This, however, contradicts (2.e).

Therefore \( v_n \not\to^2 v_1 \).

As the cases 1 and 2 takes us to a contradiction, we conclude that \( v_n \to^3 v_1 \). Thus, \( \zeta^-(v_1) = \{2, 3\} \) (see (6)), contradicting the condition 2 of the Theorem 2.1, because \( v_1 \to^3 v_n \) (by (7)).

Therefore, \( F(\gamma \cap \text{Sym}(\mathcal{C}(T))) \neq \emptyset \).

\( \square \)

Remark 2.2. The conditions on Theorem 2.1 do not imply, and they are not implicated by the conditions considered in the Theorems 1.1, 1.2, 1.3, 1.4 and 1.5.

Observe that \( T_i \) satisfies the hypothesis of the Theorem 2.1, for each \( i \in \{1,2\} \).

- \( T_1 \) does not fulfill the hypothesis of the Theorem 1.2, since \( T_1 \) contains a cycle directed of length 4 which is not quasi-monochromatic.

- \( T_1 \) does not fulfill the hypothesis of the Theorem 1.4, since \( T_1 \) contains a cycle directed of length 3 which is not \( \mathcal{C}(T_1) \)-monochromatic and a cycle directed of length 4 which is not \( \mathcal{C}(T_1) \)-monochromatic.
• $T_1$ does not fulfill the hypothesis of the Theorem 1.3, since $T_1$ contains a cycle directed of length 3 which is not $\mathcal{C}(T_1)$-monochromatic.

• $T_1$ does not fulfill the hypothesis of the Theorem 1.5, since $T_1$ contains a vertex $v$ such that $|\zeta^-(v)| = 3$.

• $T_2$ does not fulfill the hypothesis of the Theorem 1.1, since $T_2$ contains a transitive tournament of order 3 whose arcs are coloured with three distinct colours.

On the other hand, $T_j$ for every $j \in \{3,4,5,6\}$ does not fulfill the hypothesis of the Theorem 2.1, since $T_j$ contains a vertex $v$ such that $|\zeta^-(v)| = 2$ and $\zeta^-(v) \cap \zeta^+(v) \neq \emptyset$; and $T_7$ does not fulfill the hypothesis of the Theorem 2.1, since $T_7$ contains a vertex $z$ such that $|\zeta^-(z)| = 3$.

• $T_3$ fulfills the hypothesis of the Theorem 1.1, since $T_3$ doesn’t contain 3-coloured tournaments with three vertices.

• $T_4$ fulfills the hypothesis of the Theorem 1.2, since in $T_4$ each directed cycle of length at most 4 is a quasi-monochromatic cycle.

• $T_7$ fulfills the hypothesis of the Theorem 1.3, since in $T_7$ each directed cycle of length 3 is $\mathcal{C}(T_7)$-monochromatic.
• $T_6$ fulfills the hypothesis of the Theorem 1.5, since $T_6$ is a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic, and for each $v \in V(T_6)$ we have $|\zeta(v)| \leq 2$.

• $T_5$ fulfills the hypothesis of the Theorem 1.4, since in $T_5$ each tournament $T' \subseteq T_5$ such that $T'$ contains no directed cycle of length 3 satisfies that $\mathcal{C}(T')$ is a kernel-perfect digraph and each directed cycle of length 3 contained in $T_5$ is $\mathcal{C}(T_5)$-monochromatic.

3 Quasi-Tournaments

In [5] Galeana-Sánchez and García-Ruvalcaba proved the following Theorem 3.1.

**Theorem 3.1.** Let $D$ be an $m$-coloured digraph resulting from the deletion of the single arc $(x, y)$ from some $m$-coloured tournament. If $D$ does not have $C_3$ or $T_3$, then $\mathcal{C}(D)$ is a kernel-perfect digraph.

In order to prove our main result of this section we need the following Lemma 3.2.

**Lemma 3.2.** Let $D$ be an $m$-coloured digraph, with $m \geq 4$, resulting from the deletion of the single arc $(x, y)$ from some $m$-coloured tournament. If $|\zeta(v)| \leq 2$ for each vertex $v \in V(D)$, then $D$ does not contain $C_3$ or $T_3$.

**Proof.** Observe that for any two vertices adyacentes $\{u, v\} \subseteq V(D)$ $(u \neq v)$, $\zeta(u) \cap \zeta(v) \neq \emptyset$, since as there exists an arc between them, the colour of this arc belongs to $\zeta(u) \cap \zeta(v)$.

Assume, to the contrary, that there exists a subtournament of order 3, with vertices, say $\{u, v, w\}$, which is $T_3$ or $C_3$. We may assume, without loss of generality, that: the arc between $u$ and $v$ is coloured 1, the arc between $v$ and $w$ is coloured 2 and the arc between $w$ and $u$ is coloured 3. Thus, $\zeta(u) = \{1, 3\}$, $\zeta(v) = \{1, 2\}$ and $\zeta(w) = \{2, 3\}$. On the other hand, since $m \geq 4$,
there exists an arc \((z, h) \in F(D)\) such that has colour 4. As \(x\) and \(y\) are the only two nonadjacent vertices of \(D\), it follows that \(z \notin \{x, y\}\) or \(h \notin \{x, y\}\). Without loss of generality let us suppose that \(z \notin \{x, y\}\). Since \(\zeta(z) \cap \zeta(u) \neq \emptyset\) and \(\zeta(z) \cap \zeta(v) \neq \emptyset\), it follows that \(1 \in \zeta(z)\). This implies that \(\zeta(z) = \{1, 4\}\). Thus \(\zeta(z) \cap \zeta(w) = \emptyset\), which is impossible.

The following result is a direct consequence from both the Theorem 3.1 and Lemma 3.2.

**Theorem 3.3.** Let \(D\) be an \(m\)-coloured digraph, with \(m \geq 4\), resulting from the deletion of the single arc \((x, y)\) from some \(m\)-coloured tournament. If \(|\zeta(v)| \leq 2\) for each vertex \(v \in V(D)\), then \(C(D)\) is a kernel-perfect digraph.

**Remark 3.4.** If in the Theorem 3.3 we have \(m = 3\), then the result does not hold (\(D\) does not have kernel by monochromatic paths), as is shown by the following example, see figure 5. Moreover, this example shows that: if \(D\) is a 3-coloured quasi-tournament which does not contain \(C_3\), then \(D\) may does not have kernel by monochromatic paths.

![Figure 5](image)

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References


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