On the Toughness of Graphs

B. Salehian

Faculty of Mathematical Science
Damghan University of Basic Science
P.O. Box 36715-364, Damghan, Iran
bSalehian@dubs.ac.ir

Abstract

The toughness of a graph is the graph’s vulnerability to having vertices removed. For example, if vertices of the graph represent homes, and the edge of the graph represent telephone lines connecting them, then the toughness measures how badly telephone communication can be broken down by relatively few lightening strikes. In this paper the maximum networks are obtained with prescribed order and toughness.

Mathematics Subject Classification: 05C40

Keywords: toughness, nonlinear programming, vulnerability

1 Introduction and Preliminaries

Throughout of this paper, a graph always means a simple connected graph with vertex set and edge set . The toughness of a graph is an invariant first introduced by Chavátal [1]. He observed some relationships between this parameter and the existence of Hamiltonian cycles or k-factors. The toughness is an interesting invariant in graph theory and analysis of the vulnerability of a communication network to disruption. The original approach to toughness is as follows. Let $G$ be a graph and $t$ be a real number such that the implication

$$\omega(G - S) > 1 \Rightarrow |S| \geq t\omega(G - S)$$

holds for each set $S$ of points $G$. We use $\omega(H)$ to denote the number of components of the graph $H$. Then $G$ is said to be $t$-tough . If $G$ is not complete, then there is a largest $t$ such that $G$ is $t$-tough ; this number will be called the toughness of $G$ and will denoted by $t(G)$. Since a complete graph has no cutest $S$, we set $t(K_p) = \infty$ for all $p \geq 1$. A similar definition is as follows :

$$t(G) = \min_{S \subseteq V(G)} \left\{ \frac{|S|}{\omega(G - S)} : \omega(G - S) > 1 \right\}$$

If $S$ be subset of $G$ , such that $t(G) = \frac{|S|}{\omega(G - S)}$ , then we say $S$ is a $t$-set of $G$. 
2 Main Results

In this section, we give the maximum network with given order and toughness \( t \).

**Theorem 2.1.** Let \( p \) be a number of vertices, \( S \) be a \( t \)-set and \( t(G) = \frac{|S|}{\omega(G-S)} = \frac{b}{a} \) then,

\[
\max |E(G)| = \begin{cases} 
\left( \frac{p - a + 1}{2} \right) + b(a - 1) & \text{if } 2n - m \geq 1 \\
\left( \frac{p - ma + 1}{2} \right) + mb(ma - 1) & \text{if } 2n - m < 1 
\end{cases}
\]

where \( m = \left\lfloor \frac{p}{a+b} \right\rfloor \) and \( n = \frac{1}{2a} + \frac{p}{a+2b} \).

**Proof.** Let \( S \subseteq V(G) \) be a \( t \)-subset \( G \), then \( t = \frac{|S|}{\omega(G-S)} = \frac{b}{a} \) Let \( G_1, G_2, \ldots, G_l \) be components of \( G - S \), with \( |G_i| = p_i, i = 1, 2, \ldots, l \). Likewise, let \( |S| = x \geq 1 \). Thus \( \sum_{i=1}^{l} p_i + x = p \) so, we have \( t = \frac{b}{a} = \frac{p}{l-1} \). If we want the number of edges of a graph \( G \), \(|E(G)|\) to achieve the maximum, the following statement must be hold:

(I) \( G[S] \) is a complete sub-graph of \( G \).

(II) All \( G_i(i = 1, 2, \ldots, l) \) are complete sub-graphs of \( G \).

(III) All vertices in \( S \) must be adjacent to all vertices in \( G_i(i = 1, 2, \ldots, l) \).

That is, the structure of \( G \) must be similar to structure of complete graph. If the above three conditions are satisfied, let

\[
f(p_1, p_2, \ldots, p_l, x) = \sum_{i=1}^{l} \left( \frac{p_i}{2} \right) + \left( \frac{x}{2} \right) + x \sum_{i=1}^{l} p_i
\]

\[
= \frac{1}{2} \sum_{i=1}^{l} p_i^2 - \frac{1}{2} \sum_{i=1}^{l} p_i + \left( \frac{x}{2} \right) + x \sum_{i=1}^{l} p_i
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{l} p_i \right)^2 + \left( x - \frac{1}{2} \right) \sum_{i=1}^{l} p_i + \left( \frac{x}{2} \right) - \sum_{1 \leq i < j \leq l} p_i p_j
\]

\[
= \frac{1}{2} (p - x)^2 + (x - \frac{1}{2})(p - x) + \left( \frac{x}{2} \right) - \sum_{1 \leq i < j \leq l} p_i p_j \quad \text{To get the maximum value of } f(p_1, p_2, \ldots, p_l, x) \quad \text{it is necessary to make} \quad \sum_{1 \leq i < j \leq l} p_i p_j
\]

And it is easy to see that \( 1 \leq p_i \leq p - x - (l - 1)(i = 1, 2, \ldots, l) \) Now let us determine the minimum value of \( \sum_{1 \leq i < j \leq l} p_i p_j \).
i.e. solve the following nonlinear programming : \( \min g(N) = \sum_{1 \leq i < j \leq l} p_i p_j \) such that

\[
\begin{align*}
1 & \leq p_i \leq p - x - (l - 1) \\
i & = 1, 2, \ldots, l \\
\sum_{i=1}^{l} p_i & = p - x \\
\text{where} \\
p_i & \in \mathbb{Z}
\end{align*}
\]

\( N = (p_1, p_2, \ldots, p_l) \), and \( \mathbb{Z} \) is the set of positive integers.

To solve this problem, we first suppose that \( N^o = (p_1^o, p_2^o, \ldots, p_l^o) \) is an arbitrary feasible solution of the above nonlinear integer programming. Let \( p_j^o \) be the first number larger than 1 among \( p_1^o, p_2^o, \ldots, p_l^o \), i.e.,

\[ N^o = (1, 1, \ldots, 1, p_j^o, \ldots, p_l^o) \]

Construct a new feasible solution \( N^1 = (1, 1, \ldots, 1, p_j+1^o, p_j^o-1, p_j^o+2, \ldots, p_l^o) \)

Then we have \( g(N^1) \leq g(N^o) \). Repeating the above process, we can finally get a feasible solution, \( N' = (1, 1, \ldots, 1, p - x - l + 1) \).

Since \( N^o \) is an arbitrary feasible solution, we know that \( N' \) is optimal. That’s to say, when \( p_1 = p_2 = \ldots = p_{l-1} = 1 \) and \( p_l = p - x - l + 1 \), then \( \sum_{1 \leq i < j \leq l} p_i p_j \) is minimized. Now substitute these values into \( f(p_1, p_2, \ldots, p_l, x) \) we have

\[
f(1, 1, \ldots, 1, p - x - l + 1) = f_1(x) = \left( \frac{p - x - l + 1}{2} \right) + \left( \frac{x}{2} \right) + x(p - x)
\]

On the other hand, from above we know that \( t = \frac{x}{2} = \frac{b}{a} \), and so we get

\[
f_1(x) = \left( \frac{p - x - \frac{a}{b}x + 1}{2} \right) + \left( \frac{x}{2} \right) + x(p - x)
\]

see that \( x = by \), where is a positive integer number. Easily \( y \geq 1 \), also \( p_l = p - x - \frac{a}{b}x + 1 \geq 1 \) Thus , by \( x = by \) we get, \( p - by - ay + 1 \geq 1 \). So \( y \leq \frac{p}{a+b} \). Since, \( y \) is integer number, we have \( y \leq \left\lfloor \frac{p}{a+b} \right\rfloor \) So the maximum value of \( f(x) \) can be changed into the problem of finding the maximum value of \( h(y) \)

\[
\max h(y) = \left( \frac{p - ay - by + 1}{2} \right) + \left( \frac{by}{2} \right) + pby - b^2y^2
\]

\[
= \frac{1}{2} (p - (a + b)y + 1) + (p - (a + b)y) + \frac{by(by - 1)}{2} + pby - b^2y^2
\]

\[
= \frac{1}{2} [p^2 - 2p(a + b)y + (a + b)^2y^2 + p - (a + b)y] + \frac{1}{2} [b^2y^2 - by] + \frac{1}{2} [2pby - 2b^2y^2].
\]

So

\[
h(y) = \frac{1}{2} \{(a^2 + 2ab)y^2 - (2ap + a + 2b)y + p^2 + p\}
Now we consider $y$ as a real variable, we find the $h'(y)$:

$$h'(y) = (a^2 + 2ab)y - \frac{1}{2}(2ap + a + 2b) = 0$$

So we get

$$y = \frac{2ap + 2b + a}{2(a^2 + 2ab)} = \frac{1}{2a} + \frac{p}{a + 2b}$$

So, the function $h(y)$ get it’s maximum at the point $y = 1$. So we put $y = 1$ into $h(y)$. Then we get

$$\max |E(G)| = h(1) = \left( \frac{p - a + 1}{2} \right) + b(a - 1)$$

when $n < \frac{1+m}{2}$ or $2n - m < 1$

then $h(y)$ gets maximum value when $y = m$. Now substitute $y = m$ into $h(y)$, we get

$$\max |E(G)| = h(m) = \frac{a^2 + 2ab}{2}m^2 - (2ap + a + 2b) \frac{m}{2}$$

$$= \left( \frac{p - ma + 1}{2} \right) + bm(am - 1).$$

\[\square\]

### 3 Construction and Examples

From previous section we know the size of the maximum network with prescribed order and toughness. In the following, we introduce a method for constructing such maximum network $G = (V, E)$ with order $p$ and toughness $t(G) = \frac{b}{a}$. We give some examples by using $m = \left\lfloor \frac{p}{a+b} \right\rfloor$ and $n = \frac{1}{2a} + \frac{p}{a+2b}$. When we found $m$ and $n$, we would like to know which inequality, $2n - m \geq 1$ or $2n - m < 1$, are fulfilled. If $2n - m \geq 1$, then the construction is as follows:

**Step 1)** Construct the complete graph $K_{p-m+1}$

**Step 2)** Add $a - 1$ vertices to the complete graph $K_{p-m+1}$

**Step 3)** Joined each of these $a - 1$ vertices to $b$ vertices of complete graph $K_{p-m+1}$ respectively.

If $2n - m < 1$, then construction is as follows:

**Step 1)** Construct the complete graph $K_{p-ma+1}$
Step 2) Add $ma - 1$ vertices to the complete graph $K_{p - ma + 1}$

Step 3) Joined each of these $ma - 1$ vertices to $mb$ vertices of complete graph $K_{p - ma + 1}$ respectively.

![Figure 1: Graph with maximum size](image)

**Example 1.** Construct a graph $G$ with $p = 11$, $t(G) = \frac{2}{3} = \frac{b}{a}$, then,

$$n = \frac{1}{2a} + \frac{p}{a+b} = \frac{1}{2} + \frac{11}{3+2.3} = \frac{51}{28}$$

and $m = \left\lfloor \frac{p}{a+b} \right\rfloor = \left\lfloor \frac{11}{2+3} \right\rfloor = 2$.

And $2n - m = 2 \cdot \frac{51}{28} - 2 = \frac{23}{14} > 1$. Thus we use the first method to construct a graph with $\max |E(G)| = \left( \frac{11 - 3 + 1}{2} \right) + 2 \cdot (2) = 36 + 4 = 40$, $t(G) = \frac{|S|}{\omega(G-S)} = \frac{2}{3}$, as shown in Figure 1.

![Figure 2: Graph with maximum size](image)

**Example 2.** Consider a graph $G$ with $t(G) = \frac{5}{12}$ and $p = 14$. Then $n = \frac{17}{12}$ and $m = \left\lfloor \frac{14}{2+5} \right\rfloor = 2$. Since $2n - m = 2 \cdot \frac{17}{12} - 2 = \frac{49}{6} < 1$. so we use the second method of constructing. Thus we can construct a graph with $\max |E(G)| = \left( \frac{p - ma + 1}{2} \right) + mb(ma - 1) = \left( \frac{11}{2} \right) + 10.3 = 85$ as shown in Figure 2.
References


Received: January, 2010