Transversal Lightlike Submanifolds of Indefinite Kenmotsu Manifolds

S. M. Khursheed Haider

Department of Mathematics
Jamia Millia Islamia (Central University)
New Delhi-110025, India
smkhaider@rediffmail.com

Advina

Department of Mathematics
Jamia Millia Islamia (Central University)
New Delhi-110025, India
advin.maseih@gmail.com

Mamta Thakur

Department of Mathematics
Jamia Millia Islamia (Central University)
New Delhi-110025, India
mthakur09@gmail.com

Abstract

We introduce transversal lightlike submanifolds of an indefinite Kenmotsu manifold and give examples. We investigate integrability conditions of distributions which are involved in the definition of transversal lightlike submanifolds and obtain conditions under which the induced connection is a metric connection. We also study totally contact umbilical radical transversal and transversal lightlike submanifolds of indefinite Kenmotsu manifolds and prove the existence (or non-existence) of such submanifolds in an indefinite Kenmotsu space form.

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1 Introduction

The Riemannian geometry of submanifolds is one of the most important topics of differential geometry. It is well known that Semi-Riemannian submanifold[1], have many similarities with their Riemannian case. However, the lightlike submanifolds are different since their normal vector bundle intersect with the tangent bundle making it more interesting to study these submanifolds. The lightlike submanifolds were introduced and studied by Duggal and Bejancu[6]. In [2], B. Sahin initiated the study of transversal lightlike submanifolds of an indefinite Kaehler manifold which are different from CR-lightlike[6], Screen CR[7] and generalized CR-lightlike[11] submanifolds. Recently ,Yildirim and B.Sahin[4] introduce the notion of transversal lightlike submanifolds of indefinite Sasakian manifolds and obtained many interesting results.

The growing importance of lightlike submanifolds in Mathematical Physics, especially in relativity, motivated us to study transversal lightlike submanifolds extensively. Here, we introduce and study transversal and radical transversal lightlike submanifolds of indefinite Kenmotsu manifolds. The paper is arranged as follows. In section 2, we recall definitions for indefinite Kenmotsu manifolds and give basic information on the lightlike geometry needed for this paper. Section 3 is devoted to the study of the geometry of totally contact umbilical transversal lightlike submanifolds. We also discuss the existence (or non-existence ) of transversal lightlike submanifolds in an indefinite kenmotsu space form. In section 4, we give examples of transversal lightlike submanifolds, obtain integrability conditions of distributions and give geometric conditions for the induced connection to be a metric connection. In section 5, we introduce radical transversal lightlike submanifolds, give examples of such submanifolds and study integrability of distributions.

2 Preliminaries

We follow [6] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold $M^m$ immersed in a semi-Riemannian manifold $(M^{m+n}, \overline{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric $g$ induced from $\overline{g}$ and the radical distribution $(\text{Rad}TM)$ is of rank $r$, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Reimmanian complementary distribution of $(\text{Rad} TM)$ in $TM$, i.e.,

$$TM = \text{Rad}(TM) \perp S(TM)$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complemenery vector bundle of $(\text{Rad} TM)$ in $TM^\perp$. Since for any local basis
{\xi_i} of (Rad TM), there exist a local null frame \{N_i\} of sections with values in the orthogonal complement of \(S(TM^\perp)\) in \([S(TM)]^\perp\) such that \(\bar{g}(\xi_i, N_j) = \delta_{ij}\) , it follows that there exist a lightlike transversal vector bundle \(ltr(TM)\) locally spanned by \(\{N_i\}\) [[6], pg-144]. Let \(tr(TM)\) be complementary (but not orthogonal)vector bundle to TM in \(T\tilde{M}\). Then

\[
tr(TM) = ltr(TM) \perp S(TM^\perp),
\]

Following are four subcases of a lightlike submanifold \((M, g, S(TM), S(TM^\perp))\).

Case 1: \(r\)-lightlike if \(r < \min\{m, n\}\).

Case 2: Co-isotropic if \(r = n < m\); \(S(TM^\perp) = \{0\}\).

Case 3: Isotropic if \(r = m < n\); \(S(TM) = \{0\}\).

Case 4: Totally lightlike if \(r = m = n\); \(S(TM) = \{0\} = S(TM^\perp)\).

The Gauss and Weingarten equations are

\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \tag{2.1}
\]

\[
\nabla_X U = -A_U X + \nabla_X^U U, \quad \forall X \in \Gamma(TM), U \in \Gamma(tr(TM))
\]

where \(\{\nabla_X Y, A_V X\}\) and \(\{h(X, Y), \nabla_X^U V\}\) belongs to \(\Gamma(TM)\) and \(\Gamma(tr(TM))\), respectively, \(\nabla\) and \(\nabla^t\) are linear connections on \(M\) and on the vector bundle \(tr(TM)\), respectively. Moreover, we have

\[
\nabla_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM) \tag{2.2}
\]

\[
\nabla_X N = -A_N X + \nabla_X^l N + D^s(X, N), \quad \forall N \in (ltr(TM)) \tag{2.3}
\]

\[
\nabla_X W = -A_W X + \nabla_X^s W + D^l(X, W), \quad \forall W \in \Gamma(S(TM^\perp)) \tag{2.4}
\]

Denote the projection of TM on S(TM) by \(P\). Then, by using (2.1), (2.2)-(2.4) and a metric connection \(\bar{\nabla}\), we obtain

\[
\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y) \tag{2.5}
\]

\[
\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X) \tag{2.6}
\]

From the decomposition of the tangent bundle of a lightlike submanifold, we have

\[
\nabla_X P Y = \nabla_X^s P Y + h^s(X, P Y), \tag{2.7}
\]

\[
\nabla_X \xi = -A_\xi^s X + \nabla_X^s \xi, \tag{2.8}
\]
for \( X, Y \in \Gamma(TM) \) and \( \xi \in \Gamma(Rad\ TM) \). By using above equations we obtain
\[
\bar{g}(h^l(X, PY), \xi) = g(A^*_\xi X, \overline{PY})
\]
\[
(\bar{g}(h^*(X, PY), N) = g(A^*_N X, \overline{PY})
\]
\[
\bar{g}(h^l(X, \xi), \xi) = 0, A^*_\xi \xi = 0.
\]
In general, the induced connection \( \nabla \) on \( M \) is not a metric connection. Since \( \nabla \) is a metric connection, by using (2.2) we get
\[
(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y)
\]
However, it is important to note that \( \nabla^* \) is a metric connection on \( S(TM) \).
Now we recall the Gauss equation of lightlike submanifold which is given by
\[
\overline{R}(X, Y)Z = R(X, Y)Z + A_{h^l(X, Z)}Y - A_{h^l(Y, Z)}X + A_{h^*(X, Z)}Y
\]
\[
- A_{h^*(Y, Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) + D^l(X, h^*(Y, Z))
\]
\[
- D^l(Y, h^*(X, Z)) + (\nabla_X h^*)(Y, Z) - (\nabla_Y h^*)(X, Z) + D^*(X, h^l(Y, Z) - D^*(Y, h^l(X, Z))
\]
for all \( X, Y, Z \in \Gamma(TM) \).

We now recall some basic definitions of indefinite Kenmotsu manifolds which we use later.

Let \( \overline{M} \) be a \((2m+1)\)-dimensional indefinite almost contact metric manifold with almost contact metric structure \((\phi, V, \eta, \overline{g})\), where \( \phi \) is a \((1,1)\) tensor field, \( V \) is a vector field called the characteristic vector field, \( \eta \) is a 1-form and \( \overline{g} \) is a semi-Riemannian metric on \( \overline{M} \). These tensors satisfy [15]
\[
\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon \eta(X)\eta(Y), \overline{g}(V, V) = \epsilon
\]
\[
\phi^2 X = -X + \eta(X)V, \eta(X) = \epsilon \overline{g}(X, V), \epsilon = \pm 1
\]
for all \( X, Y \in \Gamma(TM) \).

It follows that
\[
\phi V = 0
\]
\[
\eta \phi = 0, \eta(V) = 1
\]

An indefinite almost contact metric manifold \( \overline{M} \) is said to be a indefinite Kenmotsu manifold if \( \forall X, Y \in \Gamma(T\overline{M}) \)
\[
\nabla_X V = -X + \eta(X)V,
\]
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\[ (\nabla_X \phi)Y = -g(\phi X, Y)V + \epsilon \eta(Y)\phi X \]  \hfill (2.15)

Throughout this paper we assume that \( \epsilon = 1 \) without loss of generality.

Let \((M, g, S(TM), S(TM^\perp))\) be a lightlike submanifold of \((\overline{M}, \overline{g})\). For any vector field \(X\) tangent to \(M\), we put

\[ \phi X = PX + FX \]

where \(PX\) and \(FX\) are the tangential and transversal parts of \(\phi X\) respectively. Moreover \(P\) is skew symmetric on \(S(TM)\).

3 Radical transversal lightlike submanifolds

In this section, we study radical transversal lightlike submanifolds of an indefinite Kenmotsu manifold. We state the following definition for a radical transversal lightlike submanifold.

**Definition 3.1.** Let \((M, g, S(TM), S(TM^\perp))\) be a lightlike submanifold, tangent to the structure vector field \(V\), immersed in an indefinite kenmotsu manifold \((\overline{M}, \overline{g})\). We say that \(M\) is a radical transversal lightlike submanifold of \(\overline{M}\) if the following conditions are satisfied:

\[ \phi(Rad TM) = ltr(TM) \]  \hfill (3.1)

\[ \phi(S(TM)) = S(TM). \]  \hfill (3.2)

In this paper, we assume that the characteristic vector field is a spacelike vector field. If \(V\) is a timelike vector field then one can obtain similar results. But it is known that \(V\) can not be lightlike.

Hereafter, \((R^{2m+1}_q, \phi_0, V, \eta, g)\) will denote the manifold \(R^{2m+1}_q\) with its usual kenmotsu structure given by

\[ \eta = dz, V = \partial z \]

\[ \overline{g} = \eta \otimes \eta + e^{2z}(- \sum_{i=1}^{q} dx_i \otimes dx_i + dy_i \otimes dy_i + \sum_{i=q+1}^{m} dx_i \otimes dx_i + dy_i \otimes dy_i) \]

\[ \phi_0\left( \sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i) + Z \partial z \right) = \sum_{i=1}^{m} (Y_i \partial x_i - X_i \partial y_i) \]

where \((x^i, y^i, z)\) are the cartesian co-ordinates.
Example 3.2. Let $\mathcal{M} = (R^n_2, \mathcal{G})$ be a semi-Euclidean space, where $\mathcal{G}$ is of signature $(-, +, +, - , +, +, +, +)$ with respect to canonical basis
\[
\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.
\]
Consider a submanifold $M$ of $R^n_2$ defined by
\[
x^1 = y^2, x^2 = y^1, x^3 = -y^4, x^4 = y^3
\]
Then a local frame of $TM$ is given by
\[
Z_1 = e^{-z}(\partial x_1 + \partial y_2), Z_2 = e^{-z}(\partial x_2 + \partial y_1)
\]
\[
Z_3 = e^{-z}(\partial x_3 - \partial y_1), Z_4 = e^{-z}(\partial x_4 - \partial y_3)
\]
\[
Z_5 = V = \partial z
\]
Hence $(Rad \ TM) = \text{span}\{Z_1, Z_2\}$ and lightlike transversal bundle $ltr(TM)$ is spanned by
\[
N_1 = \frac{e^{-z}}{2}(\partial x_1 - \partial y_2), N_2 = \frac{e^{-z}}{2}(\partial x_2 - \partial y_1)
\]
By direct calculations, we get
\[
\phi_0(Z_1) = 2N_2, \phi_0(Z_2) = -2N_1
\]
Thus $\phi_0(Rad \ TM) = ltr(TM)$. Also $\phi_0(Z_3) = -Z_4$ implies that $\phi_0(S(TM)) = S(TM)$. Hence $M$ is a radical transversal 2-lightlike manifold.

Example 3.3. Let $M = (R^n_2, \mathcal{G})$ be a semi Euclidean space, where $\mathcal{G}$ is of signature $(-, -, +, + , - , +, +, +)$ with respect to canonical basis
\[
\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}
\]
Consider a submanifold $M$ of $R^n_2$ defined by
\[
x^1 = u^1, x^2 = u^2 \sin \theta, x^3 = u^3, x^4 = -u^4 \cos \theta
\]
\[
y^1 = u^3, y^2 = -u^4 \sin \theta, y^3 = u^1, y^4 = -u^2 \cos \theta.
\]
Then a a local frame of $TM$ is given by
\[
Z_1 = e^{-z}(\partial x_1 + \partial y_3), Z_2 = e^{-z}(\sin \theta \partial x_2 - \cos \theta \partial y_4)
\]
\[
Z_3 = e^{-z}(\partial x_3 + \partial y_1), Z_4 = e^{-z}(-\cos \theta \partial x_4 - \sin \theta \partial y_2)
\]
\[
Z_5 = V = \partial z
\]
Hence $(Rad TM) = \text{span}\{Z_1, Z_3\}$ and lightlike transversal bundle $ltr(TM)$ is spanned by
\[
N_1 = \frac{e^{-z}}{2}(\partial x_1 - \partial y_3), N_3 = \frac{e^{-z}}{2}(\partial x_3 - \partial y_1)
\]
It follows that
\[
\phi_0(Z_1) = 2N_3, \phi_0(Z_3) = -2N_1
\]
Thus $\phi_0(Rad TM) = ltr(TM)$. Also $\phi_0(Z_2) = Z_4$ implies that $\phi_0(S(TM)) = S(TM)$. Hence $M$ is a radical transversal 2-lightlike manifold.
For the non-existence of 1-lightlike radical transversal lightlike submanifold, we have:

**Proposition 3.4.** There do not exist 1-lightlike radical transversal lightlike submanifold of an indefinite kenmotsu manifold $\overline{M}$.

Proof: Let $M$ be a 1-lightlike radical transversal lightlike submanifold of an indefinite kenmotsu manifold $\overline{M}$. Then

$$(\text{Rad } TM) = \text{span}\{\xi\},$$

which implies that $ltr(TM) = \text{span}\{N\}$. Using (2.10), we have

$$g(\phi\xi, \xi) = g(\phi^2\xi, \phi\xi) + \eta(\phi\xi)\eta(\xi)$$

$$= g(-\xi + \eta(\xi)V, \phi\xi)$$

Since $V$ belongs to $S(TM)$, therefore

$$g(\phi\xi, \xi) = -g(\xi, \phi\xi),$$

which implies that $g(\phi\xi, \xi) = 0$.

On the other hand, from (3.1) we have $\phi\xi = N \in ltr(TM)$. Therefore $g(\phi\xi, \xi) = g(N, \xi) = 1$, which is a contradiction. Hence $M$ can’t be 1-lightlike radical transversal lightlike submanifold.

From the definition of isotropic or totally lightlike submanifold, we have:

**Proposition 3.5.** There exist no isotropic or totally lightlike radical transversal lightlike submanifold of an indefinite kenmotsu manifold.

**Theorem 3.6.** Let $M$ be a radical transversal lightlike submanifolds of an indefinite kenmotsu manifold $\overline{M}$. Then the distribution $S(TM^\perp)$ is invariant with respect to $\phi$.

Proof: Let $W \in \Gamma(S(TM^\perp))$ and $\xi \in \Gamma(\text{Rad } TM)$. From (2.10), we have

$$g(\phi W, \xi) = -g(W, \phi \xi) = 0 \quad (3.3)$$

$$g(\phi W, N) = -g(W, \phi N) = 0 \quad (3.4)$$

which imply that $\phi(S(TM^\perp)) \cap \text{Rad } TM = \{0\}$ and $\phi(S(TM^\perp)) \cap ltr(TM) = \{0\}$. Taking $X \in \Gamma(S(TM))$ and using (2.10), (2.11) and (2.14) we obtain

$$g(\phi W, X) = -g(W, \phi X) = 0, \quad (3.5)$$

which shows that $\phi(S(TM^\perp)) \cap S(TM) = \{0\}$. 
Thus, our assertion follows from (3.3), (3.4) and (3.5).

Let \( M \) be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold \( M \). Let \( Q \) and \( T \) be the projection morphism on \( \text{Rad} \, TM \) and \( S(TM) \), respectively. Then for \( X \in \Gamma(TM) \), we have

\[
X = TX + QX
\]

where \( TX \in \Gamma(S(TM)) \) and \( QX \in \Gamma(\text{Rad} \, TM) \).

Applying \( \phi \) to (3.6), we obtain

\[
\phi X = \phi TX + \phi QX
\]

If we put \( \phi TX = SX \) and \( \phi QX = LX \), we rewrite (3.7) as

\[
\phi X = SX + LX
\]

where \( SX \in \Gamma(S(TM)) \) and \( LX \in \Gamma(ltr(TM)) \).

Let \( M \) be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold \( M \). Then from (2.15), we have

\[
\nabla_X \phi Y - \phi \nabla_X Y = (\nabla_X \phi)Y = -g(\phi X, Y) V + \eta(Y) \phi X.
\]

Using (3.8), (2.2) and (2.3), we have

\[
-g(SX, Y)V + \eta(Y)SX + \eta(Y)LX = \nabla_X SY + \nabla_X LY - \phi(\nabla_X Y)
\]

\[
= \nabla_X SY + h^l(X, SY) + h^s(X, SY) - A_{LY}X + \nabla_X^lLY + D^s(X, LY)
\]

\[
- S \nabla_X Y - L \nabla_X Y - \phi h^l(X, Y) - \phi h^s(X, Y)
\]

Considering the tangential, lightlike transversal and screen transversal component of (3.9), we respectively get

\[
(\nabla_X S)Y = -g(SX, Y)V + \eta(Y)SX + A_{LY}X + \phi h^l(X, Y)
\]

\[
h^l(X, SY) + \nabla_X^lLY - L \nabla_X Y - \eta(Y) LX = 0
\]

\[
h^s(X, SY) + D^s(X, LY) - \phi h^s(X, Y) = 0
\]

It is known that the induced connection of a lightlike submanifold is not a metric connection. Therefore it is interesting to see under what condition the induced connection on a transversal lightlike submanifold of indefinite Kenmotsu manifolds is a metric connection. The following theorem gives the necessary and sufficient condition for the induced connection to be a metric connection.
Theorem 3.7. Let $M$ be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold $M$. Then the induced connection $\nabla$ on $M$, is a metric connection if and only if $A_{\phi Y}X$ has no component in $S(TM)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad TM)$.

Proof: We recall here the fact that the induced connection is a metric connection if and only if $\nabla_X Y \in \Gamma(Rad TM)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad TM)$[[6], Pg-161]. Let the induced connection $\nabla$ on $M$ be a metric connection. Then using (2.2) for any $Z \in \Gamma(S(TM))$, we get

$$g(\nabla_X Y, Z) = 0.$$ 

Combining the above equation together with (2.10), we obtain

$$g(\phi \nabla_X Y, \phi Z) + \eta(\nabla_X Y)\eta(Z) = 0,$$

which implies that

$$g(\nabla_X \phi Y - (\nabla_X \phi)Y, \phi Z) = 0.$$ 

By using (2.15) and (2.3), we get

$$g(A_{\phi Y} X, \phi Z) = 0.$$ 

Conversely, assume that $A_{\phi Y} X$ has no component in $S(TM)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad TM)$. Then from (2.3), we have

$$g(\nabla_X \phi Y) = 0.$$ 

By using (2.15) and (2.2), we get

$$g(\nabla_X Y, \phi Z) = 0.$$ 

This implies that $\nabla_X Y \in \Gamma(Rad TM)$, which proves our assertion.

Regarding the integrability of the distributions which are involved in the definition of a radical transversal lightlike submanifold, we have the following:

Theorem 3.8. Let $M$ be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold $M$. Then $S(TM)$ is integrable if and only if

$$h^l(X, SY) = h^l(Y, SX)$$ 

for all $X, Y \in \Gamma(S(TM))$.

Proof: By interchanging $X$ and $Y$ in (3.11), we get

$$h^l(Y, SX) - L\nabla_Y X = 0 \quad (3.13)$$ 

Combining (3.11) together with (3.13), we get

$$h^l(X, SY) - h^l(Y, SX) = L[X, Y],$$

from which our assertion follows.
Theorem 3.9. Let $M$ be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $(\text{Rad } TM)$ is integrable if and only if $A_{LY}Y = A_{LY}X$

for all $X, Y \in \Gamma(\text{Rad } TM)$.

Proof: Using (3.10), we obtain $(\nabla_X S)Y = A_{LY}X + \phi h^l(X, Y)$, which implies that

$$-S\nabla_X Y = A_{LY}X + \phi h^l(X, Y).$$  \hspace{1cm} (3.14)

By interchanging $X$ and $Y$ in (3.14), we get

$$-S\nabla_Y X = A_{LY}Y + \phi h^l(Y, X).$$  \hspace{1cm} (3.15)

Combining (3.14), (3.15) and the fact that $h^l$ is symmetric, we get

$$S[X, Y] = A_{LY}Y - A_{LY}X,$$

from which our assertion follows.

Theorem 3.10. Let $M$ be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, radical distribution defines a totally geodesic foliation on $M$, if and only if $A_{\phi Y}X$ has no component in $S(TM)$ for $X, Y \in \Gamma(\text{Rad } TM)$.

Proof: By the definition of radical transversal lightlike submanifolds, $(\text{Rad } TM)$ is a totally geodesic foliation if and only if $g(\nabla_X Y, Z) = 0$ for $X, Y \in \Gamma(\text{Rad } TM)$ and $Z \in \Gamma(S(TM))$. Using (2.2) and the fact that $\overline{\nabla}$ is a metric connecton, we get $g(\nabla_X Y, Z) = X \overline{\nabla}(Y, Z) - g(Y, \nabla_X Z)$, which implies that $g(\overline{\nabla}_X Y, Z) = -g(Y, \nabla_X Z)$. By using (2.10), (2.15) and (2.2), we obtain

$$g(\overline{\nabla}_X Y, Z) = g(\nabla_X \phi Y, Z) + \eta(Z)g(X, Y).$$

This equation together with (2.3) gives

$$g(\overline{\nabla}_X Y, Z) = -g(A_{\phi Y}X, Z)$$

from which our assertion follows.

Theorem 3.11. Let $M$ be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then screen distribution defines a totally geodesic foliation if and only if $A^*_{\phi N}X$ has no component in $S(TM)$ for $X \in \Gamma(S(TM))$ and $N \in \Gamma(\text{ltr}(TM))$.

Proof: Let $M$ be a radical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, $S(TM)$ is a totally geodesic foliation if and only if $g(\nabla_X Y, N) = 0$ for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(\text{ltr}(TM))$. Using (2.2), we get $g(\nabla_X Y, N) = g(\overline{\nabla}_X Y, N)$. From (2.10), we have $g(\overline{\nabla}_X Y, N) = -g(\phi Y, \overline{\nabla}_X \phi N)$. Using (2.2) and (2.8), we obtain

$$g(\overline{\nabla}_X Y, N) = g(A^*_{\phi N}X, \phi Y),$$

from which our assertion follows.
4 Totally contact umbilical radical transversal lightlike submanifolds

We recall here the definition of a totally umbilical lightlike submanifold of a semi-Riemannian manifold given in [12].

**Definition 4.1.** A lightlike submanifold \((M, g)\) of a semi-Riemannian manifold \((\bar{M}, \bar{g})\) is said to be totally umbilical in \(\bar{M}\) if there is a smooth transversal vector field \(H \in \Gamma(\text{tr}(TM))\) on \(M\), called the transversal curvature vector field of \(M\), such that, for all \(X, Y \in \Gamma(TM)\),

\[
h(X, Y) = H\bar{g}(X, Y)
\]

Using (2.2) and (4.1), it is easy to see that \(M\) is totally umbilical if and only if on each coordinate neighbourhood \(U\), there exist smooth vector fields \(H^l \in \Gamma(ltr(TM))\) and \(H^s \in \Gamma(S(TM^\perp))\) such that

\[
\begin{align*}
\{ & h^l(X, Y) = H^l g(X, Y), \quad D^l(X, W) = 0 \quad \forall X, Y \in \Gamma(TM) \text{ and } W \in \Gamma(S(TM^\perp)), \\
& h^s(X, Y) = H^s g(X, Y) \}
\end{align*}
\]

On the other hand, in [[14], page 374], Yano and Kon have introduced the notion of contact totally umbilical submanifold of a Sasakian manifold with a definite metric which is also valid for an indefinite Sasakian manifold. Similar to the notion of contact totally umbilical submanifold of an indefinite Sasakian manifold, we define:

**Definition 4.2.** A submanifold \(M\), tangent to the structure vector field \(V\), of an indefinite Kenmotsu manifold \(\bar{M}\) is said to be contact totally umbilical if the second fundamental form \(h\) of the submanifold \(M\) is of the form

\[
h(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha
\]

for any \(X, Y \in \Gamma(TM)\), where \(\alpha\) is a vector field transversal to \(M\).

The above definition also holds for a lightlike submanifold \(M\). For a contact totally umbilical submanifold \(M\), we have

\[
\begin{align*}
h^l(X, Y) &= [g(X, Y) - \eta(X)\eta(Y)]\alpha_L, \\
h^s(X, Y) &= [g(X, Y) - \eta(X)\eta(Y)]\alpha_S
\end{align*}
\]

where \(\alpha_s \in \Gamma(S(TM^\perp))\) and \(\alpha_L \in \Gamma(ltr(TM))\).

We now investigate the integrability of screen distribution.
Theorem 4.3. Let $M$ be a totally contact umbilical radical transversal light-like submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $\alpha_L = 0$ if and only if screen distribution is integrable.

Proof: From (2.10), (2.2) and (2.15), we obtain
\[
\overline{g}([X,Y], N) = \overline{g}(h^l(X, \phi Y), \phi N) - \overline{g}(h^l(Y, \phi X), \phi N)
\] (4.4)
for all $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. From (4.3), we have
\[
h^l(X, \phi Y) = [g(X, \phi Y)]\alpha_L
\] (4.5)
\[
h^l(Y, \phi X) = [g(Y, \phi X)]\alpha_L
\] (4.6)
Thus using (4.5) and (4.6) in (4.4) we obtain
\[
\overline{g}([X,Y], N) = 2g(Y, \phi X)g(\alpha_L, \phi N),
\] which proves our assertion.

Theorem 4.4. Let $M$ be a totally contact umbilical radical transversal light-like submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $\alpha_L = 0$ if and only if $h^*(X, \phi Y) = 0$ for $X, Y \in \Gamma(S(TM) - \{V\})$.

Proof: From (2.15), (3.10) and (2.2), we obtain
\[
-g(\phi X, Y)V = \nabla_X \phi Y + h^l(X, \phi Y) + h^*(X, \phi Y) - S\nabla_X Y - L\nabla_X Y
\] (4.7)
\[-\phi h^l(X, Y) - \phi h^*(X, Y)
\]
Taking the inner product of (4.7) with $\phi \xi$, we get
\[
g(\nabla_X \phi Y, \phi \xi) - g(\phi h^l(X, Y), \phi \xi) = 0.
\]
Using (2.7), (2.10) and (4.3) we obtain
\[
g(h^*(X, \phi Y), \phi \xi) = g(X, Y)g(\alpha_L, \xi),
\] which proves our assertion.

For the induced connection on a totally contact umbilical transversal light-like submanifold to be a metric connection, we have:

Theorem 4.5. Let $M$ be a totally contact umbilical radical transversal light-like submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then the induced connection $\nabla$ on $M$ is a metric connection if and only if $A_{\phi \xi} X = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad TM)$.
Proof: Using (2.15),(2.2),(2.3) and (3.8), we obtain

\[-A_{\phi\xi}X + \nabla^l_X \phi \xi + D^s(X, \phi \xi) = S \nabla_X \xi + L \nabla_X \xi + \phi h^l(X, \xi) + \phi h^s(X, \xi)\]

(4.8)

for \(X \in \Gamma(TM)\) and \(\xi \in \Gamma(Rad TM)\). Considering the tangential components of (4.8), we get

\[-A_{\phi\xi}X = S \nabla_X \xi + \phi h^l(X, \xi) + Bh^s(X, \xi)\]

(4.9)

Using (4.3) in (4.9), we obtain

\[S \nabla_X \xi = -A_{\phi\xi}X,\]

from which our assertion follows.

**Theorem 4.6.** Let \(M\) be totally contact umbilical radical transversal light-like submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then the radical distribution is parallel if and only if

\[A_{\phi\xi_2}\xi_1 = -g(\phi\xi_1, \xi_2)V \quad \forall \ \xi_1, \xi_2 \in \Gamma(Rad TM)\]

Proof : From (2.15), we get

\[\nabla_{\xi_1} \phi \xi_2 - \phi \nabla_{\xi_1} \xi_2 = -g(\phi\xi_1, \xi_2)V\]

\(\forall \ \xi_1, \xi_2 \in \Gamma(Rad TM)\). Using (2.2),(2.3) in the above equation and considering the tangential parts we get

\[S \nabla_{\xi_1} \xi_2 = g(\phi\xi_1, \xi_2)V - A_{\phi\xi_2}\xi_1,\]

from which our assertion follows.

**Lemma 4.7.** Let \(M\) be a totally contact umbilical radical transversal light-like submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then \(\alpha_S = 0\).

Proof : From (2.15), (2.2) and (3.10), we get

\[\nabla_X \phi X + h^l(X, \phi X) + h^s(X, \phi X) - S \nabla_X X - L \nabla_X X - \phi h^l(X, X) - \phi h^s(X, X) = 0.\]

(4.10)

for all \(X \in \Gamma(S(TM) - \{V\})\). Taking screen transversal part of (4.10), we get

\[h^s(X, \phi X) = \phi h^s(X, X)\]

(4.11)

By using (4.11) and (4.3) for \(W \in S(TM^\perp)\), we get \(g(X, X)g(\alpha_S, \phi W) = 0\). Since \(S(TM)\) is a non-degenerate, we get \(\alpha_S = 0\). This completes the proof of
the lemma.

A plane section \( \Pi \) in \( T_x \overline{M} \) of a Kenmotsu manifold \( \overline{M} \) is called a \( \Pi \)-section if it is spanned by a unit vector \( X \) orthogonal to \( V \) and \( \phi X \), where \( X \) is a non null vector field on \( \overline{M} \). The sectional curvature \( K(\Pi) \) with respect to \( \Pi \) determined by \( X \) is called a \( \phi \)-sectional curvature. If \( \overline{M} \) has a \( \phi \)-sectional curvature \( c \) which does not depend on the \( \phi \)-section at each point \( p \), then \( c \) is constant in \( \overline{M} \). Then \( \overline{M} \) is called a Kenmotsu space form and is denoted by \( \overline{M}(c) \). Then the curvature tensor \( \overline{R} \) of a Kenmotsu space form \( \overline{M}(c) \) is given by

\[
\overline{R}(X,Y)Z = ((c - 3)/4)\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y \} + ((c + 1)/4)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \overline{g}(X,Z)\eta(Y)V - \overline{g}(Y,Z)\eta(X)V + \overline{g}(\phi Y, Z)\phi X - \overline{g}(\phi X, Z)\phi Y - 2\overline{g}(\phi X, Y)\phi Z \}
\]  

for any vector fields \( X, Y \) and \( Z \) on \( \overline{M} \).

For the existence (or non-existence) of radical transversal lightlike submanifold of an indefinite Kenmotsu manifold, we need the following lemmas.

**Lemma 4.8.** Let \( M \) be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Kenmotsu manifold \( \overline{M} \). Then we have

\[
h^l(\nabla_X \phi X, \xi) = 0 \quad (4.13)
\]

for \( X \in \Gamma(S(TM) - \{V\}) \) and \( \xi \in \Gamma(Rad TM) \).

Proof: Replacing \( X \) by \( \nabla_X \phi X \) and \( Y \) by \( \xi \) in (4.3), we get

\[
h^l(\nabla_X \phi X, \xi) = g(\nabla_X \phi X, \xi)\alpha_L,
\]

Using (2.7) in the above equation, we immediately get (4.13).

**Lemma 4.9.** Let \( M \) be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Kenmotsu manifold \( \overline{M} \). Then we have

\[
g(X, \nabla_{\phi X} \xi) = -g(h^l(X, \phi X), \xi) \quad (4.14)
\]

Proof: Since \( \nabla \) is a metric connection, we have

\[
\overline{g}(\nabla_{\phi X} X, \xi) + \overline{g}(X, \nabla_{\phi X} \xi) = 0.
\]

Combining (2.2) together with the above equation, we get (4.14)

**Lemma 4.10.** Let \( M \) be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Kenmotsu manifold \( \overline{M} \). Then we have

\[
g(\phi X, \nabla_X \xi) = -g(h^l(X, \phi X), \xi).
\]

(4.15)
Proof: Since \( \nabla \) is a metric connection, we have
\[
\bar{g}(\nabla_X \phi X, \xi) + \bar{g}(\phi X, \nabla_X \xi) = 0.
\]
Using (2.2) we immediately get (4.15).

We now prove the main result of this section.

**Theorem 4.11.** There exist no totally contact umbilical radical proper transversal lightlike submanifolds in an indefinite Kenmotsu space form \( \bar{M}(c) \) with \( c \neq -1 \).

Proof: Let \( M \) be a totally contact umbilical proper radical transversal lightlike submanifold of \( \bar{M}(c) \) with \( c \neq -1 \). Using (2.9),(4.12) and (4.4), we get
\[
-(c+1)2g(\phi X, \phi X)g(\phi \xi, \xi') = \bar{g}((\nabla_X h^l)(\phi X, \xi), \xi') - \bar{g}((\nabla_{\phi X} h^l)(X, \xi), \xi')
\]
for all \( X \in \Gamma(S(TM) - \{V\}), \xi, \xi' \in (\text{Rad } TM) \), where
\[
(\nabla_X h^l)(\phi X, \xi) = \nabla_X^l h^l(\phi X, \xi) - h^l(\nabla_X \phi X, \xi) - h^l(\phi X, \nabla_X \xi)
\]
(4.17)
\[
(\nabla_{\phi X} h^l)(\phi X, \xi) = \nabla_{\phi X}^l h^l(X, \xi) - h^l(\nabla_{\phi X} X, \xi) - h^l(X, \nabla_{\phi X} \xi)
\]
(4.18)
Since \( M \) is totally contact umbilical, from (4.3) we have
\[
h^l(\phi X, \xi) = 0.
\]
(4.19)
In view of lemma 4.8, we have
\[
h^l(\nabla_X \phi X, \xi) = 0
\]
(4.20)
From (4.3), we obtain
\[
h^l(\phi X, \nabla_X \xi) = g(\phi X, \nabla_X \xi)\alpha_L.
\]
(4.21)
Using (4.19),(4.20) and (4.21) in (4.17), we get
\[
\nabla_X h^l(\phi X, \xi) = -g(\phi X, \nabla_X \xi)\alpha_L.
\]
(4.22)
On the other hand, from (4.3) we have
\[
h^l(X, \xi) = 0.
\]
(4.23)
Using (4.3) and (2.7), we get
\[ h^l(\nabla_{\phi X} X, \xi) = 0. \] (4.24)

From (4.3), we obtain
\[ h^l(X, \nabla_{\phi X} \xi) = g(X, \nabla_{\phi X} \xi) \alpha_L. \] (4.25)

Using (4.23), (4.24) and (4.25) in (4.18), we have
\[ \nabla_{\phi X} h^l(X, \xi) = -g(X, \nabla_{\phi X} \xi) \alpha_L \] (4.26)

Also, using (4.22) and (4.26) in (4.16), we get
\[ \frac{-(c+1)}{2} g(X, X) g(\phi \xi, \xi') = g(X, \nabla_{\phi X} \xi) g(\alpha_L, \xi') \] (4.27)

Using (4.14) and (4.15) in (4.27), we get
\[ \frac{-(c+1)}{2} g(X, X) g(\phi \xi, \xi') = 0. \]

where we have used the symmetry of \( h^l \).

Thus we have
\[ (c+1) g(X, X) g(\phi \xi, \xi') = 0. \]

Since \( S(TM) \) and \( (\text{Rad } TM) \oplus ltr(TM) \) are non-degenerate, we can choose a non-null vector field \( X \) and \( g(\phi \xi, \xi') \neq 0 \), so \( c = -1 \). This completes the proof of the theorem.

5 Transversal Lightlike Submanifolds

Definition 5.1. Let \( (M, g, S(TM), S(TM^\perp)) \) be a lightlike submanifold, tangent to the structure vector field \( V \), immersed in an indefinite Kenmotsu manifold \( (\overline{M}, \overline{g}) \). We say that \( M \) is a transversal lightlike submanifold of \( \overline{M} \) if the following conditions are satisfied:
\[ \phi(\text{Rad } TM) = ltr(TM) \]
\[ \phi(S(TM)) \subseteq S(TM^\perp) \]

We denote the orthogonal complementry subbundle to \( \phi(S(TM)) \) in \( S(TM^\perp) \) by \( \mu \). It is easy to see that \( \mu \) is invariant.
Proposition 5.2. There does not exist 1-lightlike transversal lightlike submanifold of indefinite Kenmotsu manifold $\overline{M}$.

Proposition 5.3. There exists no isotropic or totally lightlike transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$.

The proofs of the above propositions are similar to those given in section 3.

A transversal lightlike submanifold is called proper if $S(TM) \neq 0$ and $S(TM^\perp) \neq 0$. Let $M$ be a transversal lightlike submanifold of indefinite Kenmotsu manifold $\overline{M}$. Then definition 5.1. together with invariant $\mu$ imply that (i) $\dim(Rad\ TM) \geq 2$ and (ii) Any proper 3-dimensional lightlike submanifolds must be 2-lightlike.

Example 5.4. Let $\overline{M} = (R^9_2, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-, +, +, +, -, +, +, +, +)$ w.r.t canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Consider a submanifold $M$ of $R^9_2$ defined by

$$x^1 = y^2, x^2 = y^1, x^3 = y^4, x^4 = y^3$$

Then a local frame of $TM$ is given by

$$Z_1 = e^{-z}(\partial x_1 + \partial y_2), Z_2 = e^{-z}(\partial x_2 + \partial y_1)$$

$$Z_3 = e^{-z}(\partial x_3 + \partial y_4), Z_4 = e^{-z}(\partial x_4 + \partial y_3)$$

$$Z_5 = V = \partial z$$

Hence $(Rad\ TM) = \text{span}\{Z_1, Z_2\}$ and screen transversal bundle $S(TM^\perp)$ is spanned by

$$W_1 = e^{-z}(\partial x_3 - \partial y_4), W_2 = e^{-z}(-\partial x_4 + \partial y_3).$$

By direct calculations, we get

$$\phi_0(Z_3) = -W_2, \phi_0(Z_4) = W_1$$

and lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_2), N_2 = \frac{e^{-z}}{2}(\partial x_2 - \partial y_1),$$

such that $\phi_0(Z_1) = 2N_2$ and $\phi_0(Z_2) = -2N_1$. Hence $M$ is a transversal 2-lightlike submanifold.
Example 5.5. Let \( \mathcal{M} = (\mathbb{R}^4, \mathfrak{g}) \) be a semi-Euclidean space, where \( \mathfrak{g} \) is of signature \((-,-,+,-,-,-,+,+,+,+,+)\) w.r.t canonical basis \( \{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z\} \). Consider a submanifold \( M \) of \( \mathbb{R}^4 \) defined by
\[
\begin{align*}
x^1 &= y^3, \quad x^3 = y^1, \quad x^4 = y^4, \quad x^5 = y^5, \quad x^2 = y^2 = 0.
\end{align*}
\]
Then a local frame of \( TM \) is given by
\[
\begin{align*}
Z_1 &= e^{-z}(\partial x_1 + \partial y_3), \quad Z_2 = e^{-z}(\partial x_3 + \partial y_1), \\
Z_3 &= e^{-z}(\partial x_4 + \partial y_4), \quad Z_4 = e^{-z}(\partial x_5 + \partial y_5), \\
Z_5 &= V = \partial z.
\end{align*}
\]
Hence \( (\text{Rad} \; TM) = \text{span}\{Z_1, Z_2\} \) and lightlike transversal bundle \( \text{ltr}(TM) \) is spanned by
\[
\begin{align*}
N_1 &= \frac{e^{-z}}{2}(-\partial x_1 + \partial y_3), \\
N_2 &= \frac{e^{-z}}{2}(\partial x_3 - \partial y_1).
\end{align*}
\]
By direct computations, we obtain \( \phi_0(Z_1) = 2N_2 \), \( \phi_0(Z_2) = -2N_1 \). Therefore \( \phi_0(S(TM)) \) is spanned by
\[
\begin{align*}
W_1 &= e^{-z}(\partial x_4 - \partial y_4), \\
W_2 &= e^{-z}(\partial x_5 - \partial y_5).
\end{align*}
\]
such that \( \phi_0(Z_3) = W_1 \) and \( \phi_0(Z_4) = W_2 \). Hence \( M \) is a transversal lightlike submanifold of \( \mathcal{M} \).

Let \( M \) be a transversal lightlike submanifold of an indefinite Kenmotsu manifold \( \mathcal{M} \). Let \( Q \) and \( T \) be the projection morphism on \( (\text{Rad} \; TM) \) and \( S(TM) - \{V\} \), respectively. Then for \( X \in \Gamma(TM) \), we have
\[
X = TX + QX + \eta(X)V,
\]  
where \( TX + \eta(X)V \in \Gamma(S(TM)) \) and \( QX \in \Gamma(\text{Rad} \; TM) \). Applying \( \phi \) to (5.1), we get
\[
\phi X = \phi TX + \phi QX \tag{5.2}
\]
If we take \( \phi TX = WX \) and \( \phi QX = LX \), then (5.2) can be re-written as,
\[
\phi X = WX + LX, \tag{5.3}
\]
where \( WX \in \Gamma(S(TM^\perp)) \) and \( LX \in \Gamma(\text{ltr}(TM)) \). For \( W \in \Gamma(S(TM^\perp)) \), we can write
\[
\phi W = BW + CW, \tag{5.4}
\]
where $BW \in \Gamma(S(TM))$ and $CW \in \Gamma(\mu)$.

Let $M$ be a transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Using (2.15), (5.3), (2.2)-(2.4) and (5.4) and then comparing the tangential, screen transversal and lightlike transversal components of the resulting equation, we respectively get

\[-A_{LY}X - A_{WY}X - \phi h^t(X, Y) - Bh^s(X, Y) + g(\phi X, Y)V = 0 \quad (5.5)\]

\[D^s(X, LY) + \nabla_X^{s} WY - W \nabla_X Y - Ch^s(X, Y) - \eta(Y)WX = 0 \quad (5.6)\]

\[D^l(X, WY) + \nabla_X^{l} LY - L \nabla_X Y - \eta(Y)LX = 0. \quad (5.7)\]

for all $X, Y \in \Gamma(TM)$.

We now investigate the condition under which the distribution $(Rad TM)$ is integrable.

**Theorem 5.6.** Let $M$ be transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $(Rad TM)$ is integrable if and only if

\[D^s(X, LY) = D^s(Y, LX)\]

for all $X, Y \in \Gamma(Rad(TM))$.

**Proof:** Interchanging $X$ and $Y$ in (5.6), we get

\[D^s(Y, LX) - W \nabla_Y X - Ch^s(Y, X) = 0. \quad (5.8)\]

Using (5.6), (5.8) and symmetry of $h^s$, we get

\[W[X, Y] = D^s(X, LY) - D^s(Y, LX), \quad (5.9)\]

from which our assertion follows.

**Theorem 5.7.** Let $M$ be transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $S(TM)$ is integrable if and only if

\[D^l(X, WY) = D^l(Y, WX)\]

for $X, Y \in \Gamma(S(TM))$. 

Proof: By interchanging X and Y in (5.7), we obtain

\[ D^l(X,WY) - L\nabla_X Y = 0 \quad (5.10) \]

Using (5.7) and (5.10), we get

\[ L[X,Y] = D^l(X,WY) - D^l(Y,WX), \]

from which our assertion follows.

Using (5.8) and (5.10), one can easily have the following results.

**Corollary 5.8.** Let M be transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then \(S(TM)\) defines a totally geodesic foliation if and only if \(D^l(Y,WX) = 0\) for \(X,Y \in \Gamma(S(TM))\).

**Corollary 5.9.** Let M be transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then \((\text{Rad } TM)\) defines a totally geodesic foliation if and only if \(D^s(X,LY) = Ch^s(X,Y)\) for \(X,Y \in \Gamma(\text{Rad } TM)\).

In view of the above two corollaries, we also have:

**Corollary 5.10.** A transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\) is a lightlike product manifold if and only if \(D^l(Y,WX) = 0\) and \(D^s(X,LY) \in \Gamma(\mu)\).

**Theorem 5.11.** Let M be transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then the induced connection \(\nabla\) on M is a metric connection if and only if

\[ BD^s(X,\phi Y) = \eta(\nabla_X Y) V, \]

for \(X \in \Gamma(TM)\) and \(Y \in \Gamma(\text{Rad } TM)\).

Proof: From (2.15), we have

\[ \overline{\nabla}_X \phi Y - \phi \overline{\nabla}_X Y = -g(\phi X,Y) V. \]

Using (2.2),(2.3),(2.11)and (5.3), we get

\[
-\nabla_X Y = -WA_{\phi Y}X - LA_{\phi Y}X + \phi \nabla^l_X \phi Y + BD^s(X,\phi Y) + CD^s(X,\phi Y) - \eta(\nabla_X Y) V \\
+ h^l(X,Y) + h^s(X,Y).
\]
Considering the tangential parts of the above equation, we get

\[-\nabla_X Y = \phi \nabla^I_X \phi Y + BD^s(X, \phi Y) - \eta(\nabla_X Y)V.\]  \hspace{1cm} (5.11)

for \(X \in \Gamma(TM)\) and \(Y \in \Gamma(Rad\ TM)\). From (5.11), it follows that \(\nabla_X Y \in (Rad\ TM)\) if and only if

\[BD^s(X, \phi Y) = \eta(\nabla_X Y)V,\]

which completes the proof.

**Theorem 5.12.** Let \(M\) be a totally contact umbilical transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then the induced connection \(\nabla\) on \(M\) is a metric connection if and only if \(D^s(\xi, \phi \xi) = 0\), for \(X \in \Gamma(TM)\) and \(\xi \in \Gamma(Rad\ TM)\).

**Proof:** From (2.15), we have

\[\nabla_X \phi \xi - \phi \nabla_X \xi = 0.\]

Using (2.2),(2.3),(4.3) and (5.3), we get

\[-A_{\phi \xi} X + \nabla^I_X \phi \xi + D^s(X, \phi \xi) = L \nabla_X \xi + W \nabla_X \xi \]  \hspace{1cm} (5.12)

Considering the screen transversal parts of the above equation, we get

\[D^s(X, \phi \xi) = W \nabla_X \xi,\]

from which our assertion follows.

**Corollary 5.13.** Let \(M\) be a totally contact umbilical transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). Then \((Rad\ TM)\) is parallel if and only if \(D^s(\xi_1, \phi \xi_2) = 0\), for all \(\xi_1, \xi_2 \in \Gamma(Rad\ TM)\).

**Proof:** From (2.15), we have

\[\nabla_{\xi_1} \phi \xi_2 - \phi \nabla_{\xi_1} \xi_2 = -g(\phi \xi_1, \xi_2)V\]

Using (2.2),(2.3),(4.3) and (5.3), we get

\[-A_{\phi \xi_2} \xi_1 + \nabla^I_{\xi_1} \phi \xi_2 + D^s(\xi_1, \phi \xi_2) = L \nabla_{\xi_1} \xi_2 + W \nabla_{\xi_1} \xi_2 - g(\phi \xi_1, \xi_2)V\]

Considering the screen transversal parts of the above equation, we get

\[D^s(\xi_1, \phi \xi_2) = W \nabla_{\xi_1} \xi_2.\]

Our assertion follows from the above equation and the fact that \(\nabla_{\xi_1} \xi_2 \in (Rad\ TM)\).
Lemma 5.14. Let $M$ be a totally contact umbilical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $\alpha_L = 0$ if and only if $D^s(X, \phi \xi)$ has no component in $\phi(S(TM))$ for $X \in \Gamma(S(TM) - \{V\})$ and $\xi \in \Gamma(Rad TM)$.

Proof: Let $M$ be a totally contact umbilical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. From (2.15), we have

$$\nabla_X \phi X - \phi \nabla_X X = 0.$$ Using (2.2), (2.4), (5.3) and (5.4), we get

$$-A_{\phi X} X + \nabla^*_X \phi X + D^l(X, \phi X) = W \nabla_X X + L \nabla_X X + \phi h^l(X, X) + B h^s(X, X) + C h^s(X, X)$$

Considering the tangential parts of the above equation, we get

$$-A_{\phi X} X = \phi h^l(X, X) + B h^s(X, X)$$

(5.13)

Taking the inner product of (5.13) with $\phi \xi$, we obtain

$$g(A_{\phi X} X, \phi \xi) + g(\phi h^l(X, X), \phi \xi) = 0.$$ This implies that

$$g(D^s(X, \phi \xi), \phi X) + g(h^l(X, X), \xi) = 0.$$ (5.14)

where we have used (2.6) and (2.10). Using (4.3) in (5.14), we get

$$g(D^s(X, \phi \xi), \phi X) + g(X, X)g(\alpha_L, \xi) = 0.$$ Since $S(TM)$ is non degenerate, $\alpha_L = 0$ if and only if $D^s(X, \phi \xi)$ has no component in $\phi(S(TM))$.

Theorem 5.15. Let $M$ be a totally contact umbilical transversal lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ such that $\phi(S(TM)) = S(TM^\perp)$. Then $\alpha_s = 0$ or $\dim(S(TM)) = 1$.

Proof: Let $Z \in \Gamma(S(TM) - \{V\})$. From (5.13), we have

$$g(A_{\phi X} X, Z) = g(h^s(X, X), \phi Z).$$ (5.15)

On the other hand, using (2.5) we obtain

$$g(A_{\phi X} X, Z) = g(h^s(X, Z), \phi X).$$ (5.16)

Thus from (5.15) and (5.16), we get

$$g(h^s(X, X), \phi Z) = g(h^s(X, Z), \phi X).$$ (5.17)
Using (4.3) in (5.17), we get
\[ g(X, X)g(\alpha_s, \phi Z) = g(X, Z)g(\alpha_s, \phi X). \quad (5.18) \]

By interchanging \(X\) and \(Z\) in (5.18) and then simplifying the resulting equation we get
\[ \mathcal{g}(\alpha_s, \phi X) = \frac{g(X, Z)^2}{g(X, X)g(Z, Z)} \mathcal{g}(\alpha_s, \phi X), \quad (5.19) \]
from which our assertion follows.

From the above theorem, we have:

**Corollary 5.16.** Let \(M\) be a totally contact umbilical transversal lightlike submanifold of an indefinite Kenmotsu manifold \(\overline{M}\). If \(D^s(X, \phi X) \in \Gamma(\phi(S(TM)))\) for \(X \in \Gamma(S(TM))\), then \(\dim(S(TM)) = 1\) or \(\alpha_s = 0\).

**References**


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