Measure-Theoretical Everywhere Chaos and Equicontinuity via Furstenberg Families\footnote{Supported by National Nature Science Funds of China (10771079) and Scientific Technology Planning of Guangzhou Education Bureau (08C016)}

Hongying Wu

Department of Mathematics of Guangzhou University
Guangzhou 510006, People’s Republic of China

Department of Mathematics of Huaihua College
Huaihua 418008, People’s Republic of China
wu.hongying100@163.com

Huoyun Wang

Department of Mathematics of Guangzhou University
Guangzhou 510006, People’s Republic of China
wang.huoyun@126.com

Abstract

For an invariant measure $\mu$ in a topological dynamic system, notions of $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaos and $\mathcal{F}$-$\mu$-equicontinuity are introduced and investigated, where $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ are Furstenberg families. It is shown that a TDS $(X, f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaotic if and only if there exist $\varepsilon > 0$ and $K \subseteq X \times X$ with $\mu \times \mu(K) = 1$ such that for any $(x, y)$ of $K$, one has \{$n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon$\} $\in \mathcal{F}_1$ and \{$n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta$\} $\in \mathcal{F}_2$ for any $\delta > 0$, where Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$ are extensively compatible with $(X \times X, f \times f)$, and $\mathcal{F}_1$ is a filterdual. Moreover, it turns out that if a TDS $(X, f)$ is $\mathcal{F}$-$\mu$-equicontinuous, then it is not $k\mathcal{F}$-$\mu$-pairwise-sensitive, where Furstenberg family $\mathcal{F}$ is extensively compatible with $(X \times X, f \times f)$.

Mathematics Subject Classification: 37B05; 54H20; 37B20; 58K15

Keywords: sensitive, accessible, everywhere chaos, equicontinuity, Furstenberg family
1 Introduction

Throughout this paper a topological dynamical system (TDS for short) is a pair $(X, f)$, where $X$ is a compact metric space with a metric $d$ and $f$ is a continuous surjective map from $X$ to itself. Let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of $X$. We use $M(X, f)$ to denote the collection of all $f$-invariant Borel probability measures.

Let $\mathbb{Z}_+$ be the set of non-negative integers, and let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $F$ of $\mathcal{P}$ is called a Furstenberg family [2] if it is hereditary upwards, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Let $\mathcal{F}_{\text{inf}}$ be the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that $\mathcal{F}_{\text{inf}}$ is a Furstenberg family.

The complexity of a TDS is a centrum topic of the research since the term of chaos was introduced in 1975 by Li and Yorke [11], known as Li-Yorke chaos today. Recently, chaos has attracted much attention ([5][6][11]). People have tried to give various definitions of chaos according to their understanding, for instance, everywhere-chaos [5]; $(\mathcal{F}_1, \mathcal{F}_2)$-everywhere chaos [6]. A TDS $(X, f)$ is everywhere chaotic [5] if it is sensitive and accessible. A TDS $(X, f)$ is sensitive ([1][9]) if there is $\tau > 0$ such that for every nonempty open subset $U$ of $X$ there exist $x, y$ of $U$ and a natural number $n$ such that $d(f^n(x), f^n(y)) > \tau$. A TDS $(X, f)$ is accessible [5] if for every $\varepsilon > 0$ and for any nonempty open subsets $U$ and $V$ of $X$ there are points $x \in U, y \in V$ and a natural number $n$ such that $d(f^n(x), f^n(y)) < \varepsilon$.

Now we recall some notions which are defined via Furstenberg families.

Let $(X, f)$ be a TDS, $\mu \in M(X, f)$ and $\mathcal{B}^+_X$ be the collection of Borel subsets of $X$ with positive $\mu$-measures. Suppose that $\mathcal{F}$ is a Furstenberg family.

A TDS $(X, f)$ is said to be $\mathcal{F}$-$\mu$-sensitive [4] if there is $\varepsilon > 0$–an $\mathcal{F}$-$\mu$-sensitive constant–such that for any $A \in \mathcal{B}^+_X$ there exist $x, y \in A$ with

$$\{ n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon \} \in \mathcal{F}.$$  

A TDS $(X, f)$ is said to be $\mathcal{F}$-$\mu$-pairwise sensitive [4] if there is $\varepsilon > 0$–an $\mathcal{F}$-$\mu$-pairwise sensitive constant–such that for $\mu \times \mu$-a.e. $(x, y) \in X \times X$ with

$$\{ n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon \} \in \mathcal{F}.$$  

Note: A TDS $(X, f)$ is $\mathcal{F}$-$\mu$-pairwise sensitive, then it is $\mathcal{F}$-$\mu$-sensitive (see Proposition 3.6 in [4]).

A TDS $(X, f)$ is $\mathcal{F}$-accessible [6] if for every $\varepsilon > 0$ and for any nonempty open subsets $U$ and $V$ of $X$ there are points $x \in U, y \in V$ such that $\{ n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \varepsilon \} \in \mathcal{F}$. 


A TDS \((X, f)\) is \(\mathcal{F}\)-\textit{equicontinuous} [6] if for every \(\varepsilon > 0\) there is \(\delta > 0\) such that \(d(x, y) < \delta\) implies
\[
\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \varepsilon\} \in \mathcal{F}.
\]

Now we recall some notions which come from [13].

Let \((X, f)\) be a TDS, \(\mu \in M(X, f)\) and \(K \subset X\). We say that \(K\) is \(f\)-\textit{equicontinuous} [13] if for any \(\varepsilon > 0\) there is a \(\delta = \delta(\varepsilon) > 0\) such that when \(x, y \in K\) with \(d(x, y) < \delta\) then \(d(f^n(x), f^n(y)) \leq \varepsilon\) for all \(n \in \mathbb{Z}_+\). A TDS \((X, f)\) is \(\mu\)-\textit{equicontinuous} [13] if for any \(\tau > 0\) there is an \(f\)-equicontinuous subset \(K\) of \(X\) satisfying \(\mu(K) > 1 - \tau\). A TDS \((X, f)\) is said to be \(\mu\)-\textit{sensitive} [13] if there is an \(\varepsilon > 0\)–a \(\mu\)-sensitive constant–such that for any \(A \in \mathcal{B}_X^+\) there exist \(x, y \in A\) and a natural number \(n\) with \(d(f^n(x), f^n(y)) > \varepsilon\). In [13] it is proved that if \(\mu\) is ergodic, then a TDS \((X, f)\) is \(\mu\)-equicontinuous if and only if it is not \(\mu\)-sensitive.

Following these ideas which come from ([4] [6] [13]), we have:

**Definition 1.1** Let \((X, f)\) be a TDS and \(\mu \in M(X, f)\). Suppose that \(\mathcal{F}\) is a Furstenberg family. A TDS \((X, f)\) is said to be \(\mathcal{F}\)-\textit{\(\mu\)-pairwise accessible} if for any \(\delta > 0\) and \(\mu \times \mu\)-a.e. \((x, y) \in X \times X\) one has
\[
\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}.
\]

**Definition 1.2** Let \((X, f)\) be a TDS and \(\mu \in M(X, f)\). Suppose that \(\mathcal{F}\) is a Furstenberg family. A TDS \((X, f)\) is said to be \(\mathcal{F}\)-\textit{\(\mu\)-accessible} if for any \(\delta > 0\) and for any \(A, B \in \mathcal{B}_X^+\) there exist \(x \in A\) and \(y \in B\) such that
\[
\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}.
\]

**Definition 1.3** Let \((X, f)\) be a TDS and \(\mu \in M(X, f)\). Suppose that \(\mathcal{F}\) is a Furstenberg family. Suppose that \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are Furstenberg families. A TDS \((X, f)\) is said to be \((\mathcal{F}_1, \mathcal{F}_2)\)-\textit{\(\mu\)-everywhere chaotic} if it is \(\mathcal{F}_1\)-\(\mu\)-sensitive and \(\mathcal{F}_2\)-\(\mu\)-accessible.

**Definition 1.4** Let \((X, f)\) be a TDS and \(K \subset X\). Suppose that \(\mathcal{F}\) is a Furstenberg family. We say that \(K\) is \(\mathcal{F}\)-\textit{\(f\)-equicontinuous} if for any \(\varepsilon > 0\) there is \(\delta = \delta(\varepsilon) > 0\) such that when \(x, y \in K\) with \(d(x, y) < \delta\) then \(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \varepsilon\} \in \mathcal{F}\).

**Definition 1.5** Let \((X, f)\) be a TDS and \(\mu \in M(X, f)\). Suppose that \(\mathcal{F}\) is a Furstenberg family. \(f\) or \((X, f)\) is said to be \(\mathcal{F}\)-\textit{\(\mu\)-equicontinuous}, if for any
\(\tau > 0\) there is an \(\mathcal{F}\)-\(f\)-equicontinuous subset \(K\) of \(X\) satisfying \(\mu(K) > 1 - \tau\).

In this article \((\mathcal{F}_1, \mathcal{F}_2)\)-\(\mu\)-everywhere chaos and \(\mathcal{F}\)-\(\mu\)-equicontinuity are introduced and investigated, where \(\mu \in M(X, f)\). This article is organized as follows. In section 2, some basic notions related to Furstenberg families are introduced. In section 3, we study \((\mathcal{F}_1, \mathcal{F}_2)\)-\(\mu\)-everywhere chaos. It is shown that a TDS \((X, f)\) is \((\mathcal{F}_1, \mathcal{F}_2)\)-\(\mu\)-everywhere chaotic if and only if there exist \(\varepsilon > 0\) and \(K \subset X \times X\) with \(\mu \times \mu(K) = 1\) such that for any \((x, y)\) of \(K\), one has \(\{n \in \mathbb{Z}_+: d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_1\) and \(\{n \in \mathbb{Z}_+: d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}_2\) for any \(\delta > 0\), where Furstenberg families \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are extensively compatible with \((X \times X, f \times f)\), and \(\mathcal{F}_1\) is a filterdual. Finally, we show that if a TDS \((X, f)\) is \(\mathcal{F}\)-\(\mu\)-equicontinuous, then it is not \(k\mathcal{F}\)-\(\mu\)-pairwise-sensitive, where Furstenberg family \(\mathcal{F}\) is extensively compatible with \((X \times X, f \times f)\).

2 Preliminaries

In this section, we introduce some basic notions related to Furstenberg families (for details see [2]).

For a Furstenberg family \(\mathcal{F}\), its dual family is

\[
\mathcal{kF} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset, \forall F' \in \mathcal{F}\} = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F}\}.
\]

Clearly, if \(\mathcal{F}\) is a Furstenberg family then so is \(\mathcal{kF}\). It is easy to see that \(\mathcal{kF}_{inf}\) is the family of all cofinite subsets of \(\mathbb{Z}_+\).

For Furstenberg families \(\mathcal{F}_1\) and \(\mathcal{F}_2\), let \(\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}\). A Furstenberg family \(\mathcal{F}\) is filter if it is proper and \(\mathcal{F} \supset \mathcal{F} \cdot \mathcal{F}\). A Furstenberg family \(\mathcal{F}\) is filterdual if \(\mathcal{kF}\) is filter.

Let \(J \subset \mathbb{Z}_+\). The upper density of \(J\) is

\[
\overline{d}(J) = \limsup_{n \to \infty} \frac{\#(J \cap \{0, 1, \ldots, n-1\})}{n}
\]

where \# denotes the cardinality of the set. The lower density of \(J\) is

\[
d(J) = \liminf_{n \to \infty} \frac{\#(J \cap \{0, 1, \ldots, n-1\})}{n}.
\]

If \(\overline{d}(J) = d(J)\), then we say \(J\) has density \(d(J)\).

Let \((X, f)\) be a TDS and \(U, V \subset X\). We define the meeting time set

\[
N(U, V) = \{n \in \mathbb{Z}_+ : f^n(U) \cap V \neq \emptyset\}.
\]

In particular we have \(N(x, V) = \{n \in \mathbb{Z}_+ : f^n(x) \in V\}\) for \(x \in X\).
Let $A \subset X$ and $x \in X$. If $N(x, A) \in \mathcal{F}$, $x$ is called an $\mathcal{F}$-attaching point of $A$. The set of all $\mathcal{F}$-attaching points of $A$ is called the set of $\mathcal{F}$-attaching of $A$, denoted by $\mathcal{F}(A)$. Clearly,

$$\mathcal{F}(A) = \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} f^{-n}(A) = \bigcap_{F \in k} \bigcup_{n \in F} f^{-n}(A).$$

A Furstenberg family $\mathcal{F}$ is compatible with a system $(X, f)$ [7] if the set of $\mathcal{F}$-attaching of $U$ is a $G_\delta$ set of $X$ for each open set $U$ of $X$. For each $t \in [0, 1]$, $M(t) = \{ F \in \mathcal{F}_{inf} : d(F) \geq t \}$ is compatible with any TDS [7]. A Furstenberg family $\mathcal{F}$ is extensively compatible with the system $(X, f)$ [4] if the set of $\mathcal{F}$-attaching of $U$ is a Borel set of $X$ for each open set $U$ of $X$. Clearly, if a Furstenberg family $\mathcal{F}$ is compatible with a system $(X, f)$, then the Furstenberg family $\mathcal{F}$ is extensively compatible with the system $(X, f)$. A Furstenberg family $\mathcal{F}$ is extensively compatible with a system $(X, f)$ if and only if the set of $k\mathcal{F}$-attaching of $B$ is a Borel set of $X$ for each closed set $B$ of $X$ [4].

We shall use the following some notations.

$$X^n = \underbrace{X \times X \times \cdots \times X}_n, \quad \Delta_n(X) = \{(x, x, \cdots, x) \in X^n : x \in X\}$$

$$\overline{V}_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}, \quad V_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$$

Let $(X, f)$ be a TDS and $F \in \mathcal{B}$. A pair $(x, y) \in X \times X$ is called $F$-proximal if

$$\liminf_{F \geq n \to \infty} d(f^n(x), f^n(y)) = 0.$$ 

We denote the set of all $F$-proximal pairs by $P_F$.

Let $\mathcal{F}$ be a Furstenberg family. A pair $(x, y) \in X \times X$ is called $\mathcal{F}$-proximal if $(x, y) \in \mathcal{F}(V_\varepsilon)$ for any $\varepsilon > 0$. We denote the set of all $\mathcal{F}$–proximal pairs by $P_{\mathcal{F}}$.

Note [10]:

$$P_{\mathcal{F}} = \bigcap_{\varepsilon > 0} \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} (f \times f)^{-n}(V_\varepsilon) = \bigcap_{k=1}^\infty \bigcap_{F \in k} \bigcup_{n \in F} (f \times f)^{-n}(V_{\varepsilon_k}) = \bigcap_{k=1}^\infty \mathcal{F}(V_{\varepsilon_k}) = \bigcap_{F \in k} P_F$$
3 Measure-theoretical everywhere chaos via Furstenberg families

In this section, we study the concept of $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaos via Furstenberg families.

Let $(X, f)$ be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_2$ are Furstenberg families.

A TDS $(X, f)$ is said to be $\mathcal{F}$-$\mu$-pairwise accessible if for any $\delta > 0$ and $\mu \times \mu$-a.e. $(x, y) \in X \times X$ one has

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}.$$ 

Suppose that Furstenberg family $\mathcal{F}$ is extensively compatible with $(X \times X, f \times f)$. Then $(X, f)$ is $\mathcal{F}$-$\mu$-pairwise accessible if and only if for any $\delta > 0$ one has $\mu \times \mu(k\mathcal{F}(X \times X - V_\delta)) = 0$.

A TDS $(X, f)$ is said to be $\mathcal{F}$-$\mu$-accessible if for any $\delta > 0$ and any $A, B \in \mathcal{B}^+_X$ there exist $x \in A$ and $y \in B$ such that

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}.$$ 

Remark: Suppose that $\text{supp}(\mu) = X$. If $(X, f)$ is $\mathcal{F}$-$\mu$-accessible then it is $\mathcal{F}$-accessible.

A TDS $(X, f)$ is said to be $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaotic if it is $\mathcal{F}_1$-$\mu$-sensitive and $\mathcal{F}_2$-$\mu$-accessible.

**Lemma 3.1** [8] Suppose that $X$ and $Y$ are Hausdorff spaces. $X \times Y$ is the product topological space of $X$ and $Y$ (it is Hausdorff space too). Then $\mathcal{B}(X) \times \mathcal{B}(Y) \subset \mathcal{B}(X \times Y)$. Moreover, if both $X$ and $Y$ have countable bases, then $\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)$.

**Theorem 3.2** Let $(X, f)$ be a TDS, $\mu \in M(X, f)$. Suppose that $\mathcal{F}$ be a Furstenberg family.

(1) $(X, f)$ is $\mathcal{F}$-$\mu$-pairwise accessible.

(2) $(X, f)$ is $\mathcal{F}$-$\mu$-accessible.

Then (1) $\Rightarrow$ (2). If $\mathcal{F}$ is extensively compatible with the system $(X \times X, f \times f)$, then (1) $\Leftrightarrow$ (2).

**Proof.** (1) $\Rightarrow$ (2). Assume that $(X, f)$ is $\mathcal{F}$-$\mu$-pairwise accessible. Then for any $\delta > 0$ and for $\mu \times \mu$-a.e. $(x, y) \in X \times X$ one has $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}.$

For any $A, B \in \mathcal{B}^+_X$, then $\mu \times \mu(A \times B) = \mu(A) \times \mu(B) > 0$. Hence, There is $(x, y) \in A \times B$ such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}$. This implies $(X, f)$ is $\mathcal{F}$-$\mu$-accessible.
Assume that $\mathcal{F}$ is extensively compatible with the system $(X \times X, f \times f)$, next we show that (2) $\Rightarrow$ (1). Now assume that $(X, f)$ is $\mathcal{F}$-$\mu$-accessible. If $(X, f)$ is not $\mathcal{F}$-$\mu$-pairwise accessible, then there is $\delta > 0$ such that $\mu \times \mu (k\mathcal{F}(X \times X - V_\delta)) > 0$. By lemma 3.1, one has $k\mathcal{F}(X \times X - V_\delta) = A \times B$, where $A, B \in \mathcal{B}_X^+$. Hence for any $x \in A$ and any $y \in B$ we have $(x, y) \in k\mathcal{F}(X \times X - V_\delta)$, i.e. $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \geq \delta\} \in k\mathcal{F}$. So $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \notin \mathcal{F}$, a contradiction.

**Lemma 3.3** [4] Let $(X, f)$ be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}$ is a filterdual, and is extensively compatible with the system $(X \times X, f \times f)$. Then $(X, f)$ is $\mathcal{F}$-$\mu$-sensitive if and only if it is $\mathcal{F}$-$\mu$-pairwise sensitive.

By Theorem 3.2 and Lemma 3.3, we have:

**Theorem 3.4** Let $(X, f)$ be a TDS and $\mu \in M(X, f)$. Suppose that Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$ are extensively compatible with $(X \times X, f \times f)$.

1. $(X, f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaotic.
2. there exist $\varepsilon > 0$ and $K \subset X \times X$ with $\mu \times \mu (K) = 1$ such that for any $(x, y)$ of $K$ one has
   $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_1$ and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}_2$ for any $\delta > 0$.

Then (2) $\Rightarrow$ (1). If $\mathcal{F}_1$ is filterdual, then (1) $\Leftrightarrow$ (2).

**Proof.** (2) $\Rightarrow$ (1). Assume that there exist $\varepsilon > 0$ and $K \subset X \times X$ such that $\mu \times \mu (K) = 1$, and for any $(x, y)$ of $K$ such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_1$ and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}_2$ for any $\delta > 0$. So $(X, f)$ is $\mathcal{F}_1$-$\mu$-pairwise sensitive and $\mathcal{F}_2$-$\mu$-pairwise accessible. By Proposition 3.6 in [4] and Theorem 3.2, then $(X, f)$ is $\mathcal{F}_1$-$\mu$-sensitive and $\mathcal{F}_2$-$\mu$-accessible. Thus $(X, f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaotic.

If $\mathcal{F}_1$ is filterdual, next we show that (1) $\Rightarrow$ (2). Assume that $(X, f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-$\mu$-everywhere chaotic, then $(X, f)$ is $\mathcal{F}_1$-$\mu$-sensitive and $\mathcal{F}_2$-$\mu$-accessible. By Lemma 3.3, $(X, f)$ is $\mathcal{F}_1$-$\mu$-pairwise sensitive. So there exist $\varepsilon > 0$ and $K_1 \subset X \times X$ such that $\mu \times \mu (K_1) = 1$, and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_1$ for any $(x, y)$ of $K_1$. $(X, f)$ is $\mathcal{F}_2$-$\mu$-pairwise accessible by Theorem 3.2. So for any natural number $n$ there is $K_n \subset X \times X$ such that $\mu \times \mu (K_n) = 1$, and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \frac{1}{n}\} \in \mathcal{F}_2$ for any $(x, y)$ of $K_n$. Put $K = K_1 \cap \bigcap_{n=1}^{\infty} K_n$, then the $K$ satisfies the desired condition.

**Lemma 3.5** [4] Let $(X, f)$ be a TDS, $\mu \in M(X, f)$ and supp($\mu$) = $X$. If $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is weakly mixing, then there exist $\varepsilon > 0$ and $c > 0$ such that for $\mu \times \mu$-a.e. $(x, y) \in X \times X$ with $\underline{d}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}) = c$, where $\underline{d}(J)$ denotes the lower density of the set $J$.

**Lemma 3.6** [4] Let $(X, f)$ be a TDS, $\mu \in M(X, f)$ and supp($\mu$) = $X$. If
\( f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) is weakly mixing, then \( \mu \times \mu(P_{M_\epsilon(0)}) = 1 \). \( P_{M_\epsilon(0)} \) denotes the set of all \( M_\epsilon(0) \)-proximal pairs, and \( M_\epsilon(0) \) denotes the family of all subsets of \( \mathbb{Z}_+ \) whose lower density is non-zero.

**Example 3.7** Let \((X, f)\) be a TDS, \( \mu \in M(X, f) \) and \( \text{supp}(\mu) = X \). If \( f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) is weakly mixing, then \((X, f)\) is \((M_\epsilon(c), M_\epsilon(0))\)-\( \mu \)-everywhere chaotic, where \( M_\epsilon(c) \) denotes the family of all subsets of \( \mathbb{Z}_+ \) whose lower density is not less than \( c \), and \( M_\epsilon(0) \) denotes the family of all subsets of \( \mathbb{Z}_+ \) whose lower density is non-zero.

**Proof.** By Lemma 3.5 and Lemma 3.6, \((X, f)\) is \( M_\epsilon(c) \)-\( \mu \)-pairwise sensitive and \( M_\epsilon(0) \)-\( \mu \)-pairwise accessible. By Theorem 3.2 and Proposition 3.6 in [4], \((X, f)\) is \((M_\epsilon(c), M_\epsilon(0))\)-\( \mu \)-everywhere chaotic.

### 4 Measure-Theoretical Equicontinuity via Furstenberg families

In this section, we investigate the concept of \( \mathcal{F} \)-\( \mu \)-equicontinuity via Furstenberg families.

Let \((X, f)\) be a TDS and \( K \subset X \). Suppose that \( \mathcal{F} \) is a Furstenberg family. We say that \( K \) is \( \mathcal{F} \)-\( f \)-equicontinuous if for any \( \epsilon > 0 \) there is \( \delta = \delta(\epsilon) > 0 \) such that when \( x, y \in K \) with \( d(x, y) < \delta \) then \( \{ n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \epsilon \} \in \mathcal{F} \).

It is clear that the union of finitely many \( \mathcal{F} \)-\( f \)-equicontinuous closed sets is still \( \mathcal{F} \)-\( f \)-equicontinuous. If \( X \) itself is \( \mathcal{F} \)-\( f \)-equicontinuous, then \( f \) or \((X, f)\) is called \( \mathcal{F} \)-equicontinuous.

Let \((X, f)\) be a TDS and \( \mu \in M(X, f) \). Suppose that \( \mathcal{F} \) is a Furstenberg family. \( f \) or \((X, f)\) is said to be \( \mathcal{F} \)-\( \mu \)-equicontinuous, if for any \( \tau > 0 \) there is an \( \mathcal{F} \)-\( f \)-equicontinuous subset \( K \) of \( X \) satisfying \( \mu(K) > 1 - \tau \).

Equivalently, \( f \) is \( \mathcal{F} \)-\( \mu \)-equicontinuous if for any \( \tau > 0 \) there is a compact \( \mathcal{F} \)-\( f \)-equicontinuous subset \( K \) of \( X \) satisfying \( \mu(K) > 1 - \tau \).

Note that the union of finitely many \( \mathcal{F} \)-\( f \)-equicontinuous compact sets is still \( \mathcal{F} \)-\( f \)-equicontinuous. Hence if \( f \) is \( \mathcal{F} \)-\( \mu \)-equicontinuous, then there exists an increasing sequence \( K_1 \subset K_2 \subset K_3 \cdots \) of compact \( \mathcal{F} \)-\( f \)-equicontinuous sets of \( X \) such that \( \mu(\bigcup_{i=1}^{\infty} K_i) = 1 \). Clearly, if \((X, f)\) is \( \mathcal{F} \)-equicontinuous, then \((X, f)\) is \( \mathcal{F} \)-\( \mu \)-equicontinuous for any \( \mu \in M(X, f) \).

**Definition 4.1** Let \((X, f)\) be a TDS, \( \mu \in M(X, f) \) and \( K \subset X \). Suppose that \( \mathcal{F} \) is a Furstenberg family. \( x \in K \) is said to be an \( \mathcal{F} \)-\( \mu \)-equicontinuous point on \( K \subset X \), if for any \( \epsilon > 0 \) there is \( \delta > 0 \) such that for any \( y \in K \) with \( d(x, y) < \delta \) one has

\[
\{ n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \epsilon \} \in \mathcal{F}.
\]
Proposition 4.2 Let \((X, f)\) be a TDS. Suppose that \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are Furstenberg families and \(\mathcal{F}_1 \cdot \mathcal{F}_1 \subset \mathcal{F}_2\). If every point of closed subset \(K\) of \(X\) is \(\mathcal{F}_1\)-equicontinuous point, then \(K\) is \(\mathcal{F}_2\)-equicontinuous.

Proof. If \(K\) is not \(\mathcal{F}_2\)-equicontinuous, then there is \(\varepsilon > 0\) for any \(\delta = 1/k\) such that \(d(x_k, y_k) < 1/k\) but \(\{n \in \mathbb{Z}_+ : d(f^n(x_k), f^n(y_k)) \leq \varepsilon\} \notin \mathcal{F}_2\). Without loss of generality we may assume that \(\lim_{k \to \infty} x_k = x\), then \(\lim_{k \to \infty} y_k = x\). Since \(K\) is closed, then \(x \in K\). This implies that \(x\) is \(\mathcal{F}_1\)-equicontinuous point. Thus for the given \(\varepsilon\) there is \(\zeta > 0\) such that \(d(x, y) < \zeta\) implies \(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \varepsilon/2\} \in \mathcal{F}_1\). By the triangle inequality and \(\mathcal{F}_1 \cdot \mathcal{F}_1 \subset \mathcal{F}_2\), we have \(\{n \in \mathbb{Z}_+ : d(f^n(a), f^n(b)) \leq \varepsilon\} \in \mathcal{F}_2\) for any \(a, b\) of \(B(x, \zeta) \cap K\). Choose \(x_k\) and \(y_k\) of \(B(x, \zeta) \cap K\) then \(\{n \in \mathbb{Z}_+ : d(f^n(x_k), f^n(y_k)) \leq \varepsilon\} \in \mathcal{F}_2\), a contradiction.

Corollary 4.3 Let \((X, f)\) be a TDS. Suppose that \(\mathcal{F}\) is a filter. Then every point of closed subset \(K\) of \(X\) is \(\mathcal{F}\)-equicontinuous point if and only if \(K\) is an \(\mathcal{F}\)-equicontinuous subset of \(X\).

Let \((X, f)\) be a TDS, and \(\mathcal{C}_X^0\) be the collection of all finite open covers of \(X\). Suppose that \(U\) is a finite cover of \(X, F = \{a_1 < a_2 < a_3 < \cdots\} \in \mathcal{F}_{inf}, K \subset X\). Let \(\mathcal{C}_F(U|K) = \inf_{n=1}^{\infty} N(\vee_{i=1}^{n} f^{-a_i}(U)|K), \) where \(N(\vee_{i=1}^{n} f^{-a_i}(U)|K)\) denotes the minimum among the cardinalities of all the subsets of \(\vee_{i=1}^{n} f^{-a_i}(U)\) which cover \(K[12]\). When \(K = X\), we omit the restriction of \(K\).

Proposition 4.4 Let \((X, f)\) be a TDS and \(\mu \in M(X, f)\).

(1) \((X, f)\) is \(\mathcal{F}\)-\(\mu\)-equicontinuous.

(2) For any \(\varepsilon > 0\) and any \(\tau > 0\), there exist a compact subset \(K\) of \(X\) and \(\delta > 0\) satisfying \(\mu(K) > 1 - \tau, \) and for any \(x, y \in K\) with \(d(x, y) < \delta\) one has \(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \varepsilon\} \in \mathcal{F}\).

(3) For any \(\tau > 0\) there exist a compact subset \(K\) of \(X\) and \(F \in \mathcal{F}\) satisfying \(\mu(K) > 1 - \tau\) and \(\mathcal{C}_F(U|K) < +\infty\) for any \(U \in \mathcal{C}_X^0\).

(4) For any \(\mathcal{F} \subset \mathcal{F}_{inf}\) and \(\tau > 0\) there exist a compact subset \(K\) of \(X\) and \(F \in \mathcal{F}\) satisfying \(\mu(F) > 1 - \tau\) and \(\mathcal{C}_F(U|K) < +\infty\).

Then (1) \(\Leftrightarrow\) (2), (3) \(\Rightarrow\) (4). If \(\mathcal{F}\) is filter and \(\mathcal{F} \subset \mathcal{F}_{inf}\), then (4) \(\Rightarrow\) (2).

Proof. (1) \(\Rightarrow\) (2) and (3) \(\Rightarrow\) (4) are obvious.

(2) \(\Rightarrow\) (1). Assume that (2) holds. Given \(\tau > 0\), for any positive integer \(l\), by (2) there exist a compact subset \(K_i\) of \(X\) and \(\delta_i > 0\) satisfying \(\mu(K_i) > 1 - \frac{\tau}{l}\) and for any \(x, y \in K_i\) with \(d(x, y) < \delta_i\), one has \(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq 1/l\} \in \mathcal{F}\). Let \(K = \bigcap_{i=1}^{\infty} K_i\). Then \(K\) is a compact \(\mathcal{F}\)-equicontinuous subset of \(X\) and \(\mu(K) = 1 - \mu(\bigcup_{i=1}^{\infty} K_i^c) \geq 1 - \sum_{i=1}^{\infty} \mu(K_i^c) \geq 1 - \sum_{i=1}^{\infty} \frac{\tau}{l} = 1 - \tau\).

If \(\mathcal{F}\) is filter and \(\mathcal{F} \subset \mathcal{F}_{inf}\), next we show that (4) \(\Rightarrow\) (2). Assume that
(4) holds. For any \( \varepsilon > 0 \) and any \( \tau > 0 \), we choose a finite open cover \( U \) of \( X \) with \( \text{diam}(U) < \varepsilon/4 \). Then there exist a compact subset \( K \) of \( X \) and \( F \in \mathcal{F} \) satisfying \( \mu(K) > 1 - \tau \) and \( \mathcal{C}_F(U|K) < +\infty \). We claim that there is \( \delta > 0 \) such that for any \( x, y \in K \) with \( d(x,y) < \delta \), one has \( \{ n \in \mathbb{Z}_+: d(f^n(x), f^n(y)) \leq \varepsilon \} \in \mathcal{F} \).

If this is not the case, then for any \( \delta = 1/k \) there exist \( x_k \) and \( y_k \) of \( K \) such that \( d(x_k, y_k) < 1/k \) but \( \{ n \in \mathbb{Z}_+: d(f^n(x_k), f^n(y_k)) \leq \varepsilon \} \notin \mathcal{F} \). Without loss of generality we may assume that \( \lim_{k \to \infty} x_k = x \), then \( \lim_{k \to \infty} y_k = x \). Since \( K \) is closed, then \( x \in K \). By the triangle inequality and \( \mathcal{F} \) is filter, either \( \{ n \in \mathbb{Z}_+: d(f^n(x), f^n(x_k)) \leq \varepsilon/2 \} \notin \mathcal{F} \) or \( \{ n \in \mathbb{Z}_+: d(f^n(x), f^n(y_k)) \leq \varepsilon/2 \} \notin \mathcal{F} \) for any positive integer \( k \). That is, there exists \( x \in K \) such that for any \( \delta > 0 \) we can find \( y \in K \) satisfying \( d(x,y) < \delta \) and \( \{ n \in \mathbb{Z}_+: d(f^n(x), f^n(y)) \leq \varepsilon/2 \} \notin \mathcal{F} \). Let \( \mathcal{F} = \{ U_1, U_2, \ldots, U_k \} \) be the cover of \( X \) consisting of the closure of the elements in \( \mathcal{U} \). Clearly, \( \mathcal{U} \leq \mathcal{F} \) and so \( c := \mathcal{C}_F(\mathcal{U}|K) \leq \mathcal{C}_F(\mathcal{U}|K) < +\infty \).

Therefore, there exists a finite closed cover \( \{ K_1, K_2, \ldots, K_c \} \) of \( K \) such that \( K_i = K \cap \bigcap_{n \in \mathbb{F}} f^{-n}U_{i,t}, U_{i,t} \in \mathcal{U} \) (see [3]).

Now by definition of \( \mathcal{U} \), we know that if \( y, z \) belong to the same \( K_i \), then \( \{ n \in \mathbb{Z}_+: d(f^n(x), f^n(y)) \leq \varepsilon/2 \} \notin \mathcal{F} \). Let \( \delta_n > 0 \) converge to 0 and choose \( y_n \in K \) such that \( d(x, y_n) < \delta_n \) and \( \{ m \in \mathbb{Z}_+: d(f^m(x), f^m(y_n)) \leq \varepsilon/2 \} \notin \mathcal{F} \). By choosing a suitable subsequence, we may suppose that \( y_n \) belong to the same \( K_i \). Hence \( x \in K_i \) as \( K_i \) is closed. This implies that \( \{ m \in \mathbb{Z}_+: d(f^m(x), f^m(y_n)) \leq \varepsilon/2 \} \notin \mathcal{F} \), a contradiction.

**Theorem 4.5** Let \( (X, f) \) be a TDS and \( \mu \in M(X,f) \). Suppose that Furstenberg family \( \mathcal{F} \) is extensively compatible with \( X \times X, f \times f \). If \( f \) is \( \mathcal{F}_\mu \)-equicontinuous, then \( f \) is not \( k\mathcal{F}_\mu \)-pairwise-sensitive.

**Proof.** Assume that \( f \) is \( \mathcal{F}_\mu \)-equicontinuous. By Proposition 4.4(2), for any \( \varepsilon > 0 \) we can find a compact subset \( K \) of \( X \) and \( \delta > 0 \) satisfying \( \mu(K) > 1/2 \) and for any \( x, y \in K \) with \( d(x,y) < \delta \), one has \( \{ n \in \mathbb{Z}_+: d(f^n(x), f^n(y)) \leq \varepsilon \} \in \mathcal{F} \).

Since \( \{ B(x, \delta/3) \}_{x \in K} \) forms an open cover of \( K \), there exists \( x \in K \) such that \( \mu(A) > 0 \) with \( A := B(x, \delta/3) \cap K \). Hence for any \( (x_1, x_2) \in A \times A \), there is \( F \in \mathcal{F} \) such that \( (x_1, x_2) \in \bigcap_{n \in \mathbb{F}} f^{-n}V_\delta \), this implies that \( (x_1, x_2) \in V_\delta \). So \( \mu \times \mu(V_\delta) = \mu \times \mu(\bigcup_{n \in \mathbb{F}} f^{-n}(V_\delta)) \geq \mu \times \mu(A \times A) = \mu(A)^2 > 0 \). Note that \( \mu \times \mu(V_\delta) = 0 \) if and only if \( f \) is \( k\mathcal{F}_\mu \)-pairwise-sensitive (see [4]). Thus \( f \) is not \( k\mathcal{F}_\mu \)-pairwise-sensitive.

Let \( (X, f) \) be a TDS and \( (x_i)_{i=1}^n \in X^n \). A finite cover of \( X, \mathcal{U} = \{ U_1, U_2, \ldots, U_k \} \), is said to be admissible with respect to \( (x_i)_{i=1}^n \) if for each \( 1 \leq j \leq k \) there exists \( 1 \leq i_j \leq n \) such that \( x_{i_j} \) is not contained in the closure of \( U_j \).

**Definition 4.6** Let \( (X, f) \) be TDS, \( \mu \in M(X,f) \) and \( F \in \mathcal{F}_{inf} \). An \( n\)-
tuple \((x_i)_{i=1}^n \in X^n (n \geq 2)\) is called an \(F-\mu\)-complexity \(n\)-tuple if at least two points in \((x_i)_{i=1}^n\) are different and for any admissible finite open cover \(U\) with respect to \((x_i)_{i=1}^n\) there exists \(\tau > 0\) such that for any compact subset \(K\) of \(X\) with \(\mu(K) > 1 - \tau\), one has \(C_F(U|K) = +\infty\).

Denote by \(Com^\mu_{F,n}(X,f)\) the set of all \(F-\mu\)-complexity \(n\)-tuples.

We have the following Lemma whose proof is similar the Lemma 4.2 in [13].

**Lemma 4.7** Let \((X,f)\) be a TDS, \(\mu \in M(X,f)\) and \(U = \{U_1, U_2, \ldots, U_k\} \in C^0_X\). Let \(F = \{a_1 < a_2 < a_3 < \cdots\} \in F_{inf}\). If there exists \(\tau > 0\) such that for any compact set \(K\) of \(X\) with \(\mu(K) > 1 - \tau\) one has \(C_F(U|K) = +\infty\), then there exists \(x_i \in U_i^c\) such that \((x_i)_{i=1}^k \in Com^\mu_{F,k}(X,f)\).

**Theorem 4.8** Let \((X,f)\) be a TDS and \(\mu \in M(X,f)\). Suppose that \(F\) is a filter, and \(F \subset F_{inf}\). If there exists \(F \in F\) such that \(Com^\mu_{F,2}(X,f) = \emptyset\), then \((X,f)\) is \(\mathcal{F}-\mu\)-equicontinuous.

**Proof.** Assume that there exists \(F \in \mathcal{F}\) such that \(Com^\mu_{F,2}(X,f) = \emptyset\). If \((X,f)\) is not \(\mathcal{F}-\mu\)-equicontinuous, by Proposition 4.4(4), then there exist \(U = \{U_1, U_2, \ldots, U_k\} \in C^0_X\) and \(\tau > 0\) such that for any compact set \(K\) of \(X\) with \(\mu(K) > 1 - \tau\) one has \(C_F(U|K) = +\infty\). Thus one has \(k \geq 2\), by Lemma 4.7 there exists \(x_i \in U_i^c\) such that \((x_i)_{i=1}^k \in Com^\mu_{F,k}(X,f)\). Since \((x_i)_{i=1}^k \notin \Delta_k(X)\), there exists \(1 \leq i_1 < i_2 \leq k\) with \(x_{i_1} \neq x_{i_2}\). It is clear that \((x_{i_1}, x_{i_2}) \in Com^\mu_{F,2}(X,f)\), a contradiction.

**References**


Received: January, 2010