The Criterion of Supersolubility
for Product of Finite Groups

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Abstract

A subgroup \( H \) of a group \( G \) is said to be \( s \)-semipermutable if it is permutable with every Sylow \( p \)-subgroup of \( G \) with \((p, |H|) = 1\). In this paper, we give a condition under which the product of two supersoluble groups is supersoluble by using \( s \)-semipermutable subgroups.

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1 Introduction

Throughout this paper, all groups considered are finite. The notations and terminologies are standard, as in [4].

The well-known Fitting’s Theorem says that the product of any two normal nilpotent subgroups is nilpotent as well. However, supersoluble groups do not have this property. It is natural to ask under what additional conditions

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the product of two supersoluble groups will still be supersoluble. For this question, people have obtained many interesting results. For example, Baer\cite{1} proved that the product \( G = AB \) of two normal supersoluble groups \( A \) and \( B \) is supersoluble if \( G' \) is nilpotent. In this paper, we will give a condition under which the product of two supersoluble groups is supersoluble by using \( s \)-semipermutable subgroups.

**Definition** (\cite{5}). A subgroup \( H \) of a group \( G \) is said to be \( s \)-semipermutable if it is permutable with every Sylow \( p \)-subgroup of \( G \) with \( (p, |H|) = 1 \).

By using the above concept, we prove the following new result.

**Theorem.** Let \( G = AB \) be the product of supersoluble subgroups \( A \) and \( B \). If all primary cyclic subgroups of \( A \) and \( B \) are \( s \)-semipermutable in \( G \), then \( G \) is supersoluble.

## 2 Preliminaries

For the sake of convenience, we list the following known results which are useful in the sequel.

**Lemma 2.1.** Let \( H \) be a proper subgroup of a group \( G \). Then \( HH^x \neq G \), for all \( x \in G \).

**Lemma 2.2.** Let \( G = AB \). Let \( A_p, B_p \) and \( G_p \) be sylow \( p \)-subgroups of \( A,B \) and \( G \) respectively. Then there are elements \( x, y \in G \) such that \( G_p^x = A_p B_p^y \).

**Lemma 2.3.** Let \( G \) be a group. Then the following statements hold:

1. If \( G \) is supersoluble, then \( G' \subseteq F(G) \), and \( G \) is \( p \)-closed for the largest prime divisor \( p \) of \( |G| \);
2. If \( L \leq G \) and \( G/\Phi(L) \) is supersoluble, then \( G \) is supersoluble;
3. \( G \) is supersoluble if and only if \( |G : M| \) is a prime for every maximal subgroup \( M \) of \( G \).

**Lemma 2.4** (\cite{3}). Let \( G = AB \) be the product of its subgroups \( A, B \). If \( L \leq A \) and \( L \subseteq B \), then \( L \subseteq B_G \).

**Lemma 2.5** (\cite{5}). Let \( A \) and \( B \) be both \( s \)-semipermutable in \( G \), then the following statements hold:

1. If \( A \leq H \leq G \), then \( A \) is \( s \)-semipermutable in \( H \);
Criterion of supersolubility

(2) If \( A \) is a \( p \)-group and \( N \trianglelefteq G \), then \( AN/N \) is \( s \)-semipermutable in \( G/N \);
(3) If \( AB = BA \), then \( AB \) is \( s \)-semipermutable in \( G \).

Lemma 2.6. Let \( H \) be a Sylow 2-subgroup of \( G \), and \( H \) be \( s \)-semipermutable in \( G \), then \( G \) is soluble.

Proof. By [2, a generalization for Schur-Zassenhause’s Theorem], \( H \) has a complement \( L \) in \( G \). That is, \( G = HL, H \cap L = 1 \). Since \( L \) is a group of order odd, \( L \) is soluble. Let \( L_{p_1}, L_{p_2}, \ldots, L_{p_m} \) be a Sylow basis of \( L \). By hypothesis \( HL_{p_i} = L_{p_i}H \) and \( L_{p_i}L_{p_j} = L_{p_j}L_{p_i} \) for \( i, j = 1, 2, \ldots, m \). This implies \( G \) has a Sylow basis. So \( G \) is soluble.

3 The Proof of Theorem.

Equipped with the above concept and lemmas, we now prove the Theorem stated in Section 1.

Proof of Theorem. Assume this theorem is false and let \( G \) be a counterexample of minimal order. Then \( G \neq A, B \). We proceed the proof as follows:

(a) \( G \) is soluble.

By Lemma 2.2, we have that \( G_2 = A_2B_2 \), where \( A_2, B_2 \) and \( G_2 \) are Sylow 2-subgroups of \( A, B \) and \( G \), respectively. Without loss of generality, let \( 1 \neq K < \cdot A_2 \). Then there exists \( a \in A_2 \setminus K \) such that \( A_2 = K \langle a \rangle \). Similarly, If \( 1 \neq K_1 < \cdot K \), then \( K = K_1 \langle a_1 \rangle, a_1 \in K \setminus K_1 \). Using the same argument as above until \( K_s = K_{s+1} \langle a_{s+1} \rangle \), where \( K_{s+1} < \cdot K_s, a_{s+1} \in K_s \setminus K_{s+1}, |K_{s+1}| = 2 \).
By hypothesis \( K_{s+1} \) and \( a_{s+1} \) are both \( s \)-semipermutable in \( G \). By Lemma 2.5(3), we have that \( K_{s+1} \langle a_{s+1} \rangle = K_s \) is also \( s \)-semipermutable in \( G \). Similarly, we can prove that \( A_2 \) is \( s \)-permutable in \( G \). Using the same argument as above, we have that \( B_2 \) is \( s \)-semipermutable in \( G \). By Lemma 2.5(3) again, \( G_2 \) is \( s \)-semipermutable in \( G \). Thus \( G \) is soluble by Lemma 2.6.

(b) If \( M \) is a proper subgroup of \( G \) and either \( A \subseteq M \) or \( B \subseteq M \), then \( M \) is supersoluble.

Assume that \( A \subseteq M \). Then \( M = M \cap AB = A(M \cap B) \). Let \( T \) and \( K \) be primary cyclic subgroups of \( A \) and \( M \cap B \), respectively. By hypothesis and Lemma 2.5(1), \( T \) and \( K \) are both \( s \)-semipermutable in \( M \). This shows that \( M \) is supersoluble since \( |M| < |G| \).
(c) $G/N$ is a supersoluble group for any non-identity normal subgroup $N$ of $G$.

Obviously, $G/N = (AN/N)(BN/N)$, $AN/N \simeq A/(A \cap N)$ and $BN/N \simeq B/(B \cap N)$ are supersoluble. Let $\langle a \rangle N/N$ be a primary cyclic subgroup of $AN/N$, where $a \in A$. By Lemma 2.5(2), $\langle a \rangle N/N$ is $s$-semipermutable in $G/N$. Analogously, every primary cyclic subgroup of $BN/N$ is $s$-semipermutable in $G/N$. Since $|G/N| < |G|$, $G/N$ is supersoluble.

(d) $G$ has the only minimal normal subgroup $H = O_p(G) = C_G(H)$, for some prime $p$, and $G = [H]M$, where $M$ is a maximal subgroup of $G$ with $O_p(M) = 1$ and $|H| \neq p$.

Let $H$ be a minimal normal subgroup of $G$. Since the class of all supersoluble groups is a saturated formation, in view of (c), $H$ is the only minimal normal subgroup of $G$ and $H \not\leq \Phi(G)$. Let $M$ be a maximal subgroup of $G$ not containing $H$ and $C = C_G(H)$. Then $C = C \cap HM = H(C \cap M)$. Since $H$ is abelian, $C \cap M \leq G$ and so $C \cap M = 1$. This shows that $H = C_G(H) = O_p(G)$ and $M \simeq G/H$ is a supersoluble group with $O_p(M) = 1$. By (c) and the choice of $G$, we have that $|H| \neq p$.

(e) $p$ is the largest prime divisor of $|G|$.

Assume that $q$ is the largest prime divisor of $|G|$ and $q \neq p$. Let $T_1$ and $T_2$ be maximal subgroups of $G$ such that $A \leq T_1, B \leq T_2$. Then by (b), $T_1, T_2$ are supersoluble. Since $G = AB = T_1T_2$, by Lemma 2.1, $T_1 \not\leq T_2$ for all $x \in G$. Hence by [4, Corollary 2.4.4], $(T_1)_G \not\leq (T_2)_G$ and so either $H \subseteq T_1$ or $H \subseteq T_2$. Without lose of generality, assume that $H \subseteq T_1$. Let $G_q$ be a Sylow $q$-subgroup of $G$. Since $G/H$ is supersoluble and $T_1/H$ is maximal in $G/H$, $|G : T_1| = |G/H : T_1/H|$ is a prime by Lemma 2.3(3). If $q |||T_1||$, then there exists a non-trivial Sylow $q$-subgroup $Q$ of $T_1$. By Lemma 2.3(1), $Q \leq T_1$.

It follows that $Q \subseteq C_G(H) = H$, a contradiction. Hence $q \nmid |A|$ and $q \mid |B|$. This implies that the Sylow $q$-subgroup $B_q$ of $B$ is also a Sylow $q$-subgroup of $G$. Let $x$ be an arbitrary element in $A$. If $\langle x \rangle$ is a subgroup of primary order, then by hypothesis, $\langle x \rangle B_q = B_q \langle x \rangle$. If $o(x) = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}$ and $r > 1$, then let $x_1 = x^{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_{r-1}^{\alpha_{r-1}}}$ and $y_1 = x^{p_2^{\alpha_2}}$, we have that $\langle x_1 \rangle \langle y_1 \rangle = \langle x \rangle$. If $r - 1 \geq 2$, then we let $x_2 = y_1^{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_{r-2}^{\alpha_{r-2}}}$, $y_2 = y_1^{p_{r-1}^{\alpha_{r-1}}}$, we have that $\langle x_2 \rangle \langle y_2 \rangle = \langle y_1 \rangle$. It follows that $\langle x \rangle = \langle x_1 \rangle \langle x_2 \rangle \langle y_2 \rangle$. If $r - 2 \geq 2$, then using the same argument as above until $\langle x \rangle = \langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_{r-1} \rangle \langle y_{r-1} \rangle$, where $o(y_{r-1}) = p_1^{\alpha_1}$ and $o(x_i) = p_2^{\alpha_2}\ldots p_{r-1}^{\alpha_{r-1}}$. 

Na Tang
Criterion of supersolubility

By hypothesis, \( \langle x \rangle B_q = B_q \langle x \rangle \). By the arbitrary choice of \( x \), we have that \( AB_q = B_q A \). Thus \( G = AB_q B \). By Lemma 2.4, \( B_q \leq (AB_q) G \leq AB_q \leq G \). If \( AB_q < G \), then \( AB_q \) is supersoluble by (b). It follows that \( B_q \trianglelefteq AB_q \) and so \( B_q \subseteq C_G(H) = H \). This contradiction shows that \( AB_q = G \). Let \( r \) be the largest prime divisor of \( |A| \) and \( A_1 \) a minimal normal subgroup of \( A \). Then \( A_1 B_q = B_q A_1 \) and \( |A_1| = r \). Thus \( G = AB = AA_1 B_q \) and by Lemma 2.4, \( A_1 \leq (A_1 B_q) G \leq A_1 B_q \leq G \). Clearly, \( H \leq A_1 B_q \leq G \). So \( p = r \) and \( |H| = |A_1| = p \). This contradicts to (d). So (e) holds.

(f) \( H \) is a Sylow \( p \)-subgroup of \( G \).

By (c) and (e), \( G/H \) is supersoluble and \( p \) is the largest prime divisor of \( |G| \). It follows that the Sylow \( p \)-subgroup of \( G/H \) is normal in \( G/H \). Since \( H = O_p(G) \), \( O_p(G/H) = 1 \). Thus (f) holds.

(g) To complete the proof.

Since \( G = AB \) and \( H \) is the normal Sylow \( p \)-subgroup of \( G \), either \( H \cap A \neq 1 \) or \( H \cap B \neq 1 \). Let \( H \cap A = A_p \neq 1 \) and \( Z_p \) a minimal subgroup of \( A \). By hypothesis, \( D = Z_p G_{p'} = G_{p'} Z_p \) where \( G_{p'} \) is a Hall \( p' \)-subgroup of \( G \). Since \( Z_p = H \cap Z_p G_{p'} \), \( Z_p \trianglelefteq HG_{p'} = G \). Hence \( Z_p = H \). This contradiction completes the proof.

References


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