A Fuzzy Sequential Decision Procedure Applied to Completely Randomized Design of Experiments

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Abstract

In this article, we have considered the fuzzification of a sequential inference procedure to be used in case of a Completely Randomized Design. We studied here, in particular, the sequential procedure developed by Ray (1956) for testing hypotheses regarding the possible differences between the means of different groups in the case of a one-way classification by groups, for fuzzy information. Accordingly a fuzzy sequential testing procedure has been developed. This is followed the application of the fuzzy procedure to a numerical problem and a comparison is made between the fuzzy and non-fuzzy inference procedure.

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1.INTRODUCTION

Existing inference methods have been using the concept of fuzziness to accommodate situations in which random events take place in fuzzy circumstances, since the middle of 1980’s. Goswami et al (1997) studied the fuzzification of certain experimental designs in the
analysis of variance, for the first time. That was a case of fixed-sample analysis (non-sequential). But no work on sequential design of experiments with fuzzy information have been made so far. In the field of fuzzy sequential analysis, as a pioneer work, Talukdar and Baruah (2007) have fuzzified the Wald’s Sequential Probability Ratio Test. In this present article, our aim is to develop a fuzzy sequential decision procedure to be used in a Sequential Completely Randomized Design. For the purpose of fuzzification, we consider the sequential method, suggested by Ray (1956).

After some fundamental works due to Barnard (1952) and Cox (1952), Johnson (1953) has derived a procedure for applying sequential test to the cases of general linear hypothesis. Hoel (1955) obtains a similar test by a rather different method.

The general linear hypothesis underlies a number of common analysis of variance situations. In order to accommodate such situations, Arnold (1951) suggested a sequential procedure for testing two composite hypotheses, based on a likelihood ratio, which is approximated with the help of Confluent hyper geometric function. Ray (1956), considered the applications of this general method to the two special cases of design of experiments, namely a) One way classification by groups and  b) Randomized blocks. In the course of construction of the test criterion, a likelihood ratio, in this case, to be used in the sequential testing procedure, the author has assumed that the non-centrality element \( \lambda = N\delta \), where \( \delta \) does not depend on \( N \) (the total no. of observations) and \( H_1 \), the alternative hypothesis is defined in terms of \( \delta \). Moreover, several approximations to the Confluent hyper geometric function have been made in developing the likelihood ratio as it was done in the general procedure. Again our problem of testing for the equality of means of several groups is not a problem of testing one simple hypothesis against another. It is a case of testing composite hypotheses. As is well known, the cumulative sums for the case of the simple hypotheses of a sequential decision procedure, we end up using Wald’s identity, that is geared to the problem of random walk. However, for the case of composite hypotheses, the principle of random walk is violated. Hence, in this situation, unlike in the case of testing for simple hypotheses, exact expressions for the Operating Characteristic function and the Average Sample Number function are not available. We have therefore to be content with certain lower and upper bounds only. Accordingly in the present article we will not get results for fuzzy Operating Characteristic function and fuzzy Average Sample Number function.

For fuzzification of the procedure given by Ray (1956), we consider the observations as well as the parameters involved in the test procedure to be fuzzy triangular numbers. Following the works of Goswami and Baruah (1997), Talukdar and Baruah (2007) and applying the theory of Kaufmann and Gupta (1984), we finally arrive at the required fuzzy sequential decisions rules. This fuzzy sequential procedure is then compared with the original non fuzzy procedure with the help of a numerical example.

In the following sections, we start with the description of the sequential test procedure forwarded by Ray (1956).
2. SEQUENTIAL TEST PROCEDURE APPLIED TO COMPLETELY RANDOMIZED DESIGN

This is the simplest design in the analysis of variance. The data, here are arranged in K groups and it is desired to reach conclusion about possible differences between the means of the different groups. The sequential procedure, considered here for fuzzification, consists of taking an equal number of observations from each group at each stage in the experiment. The theoretical model can be expressed in the following form:

\[ X_n = a + b_t + Z_n \quad (t = 1,2, \ldots, K; i = 1,2, \ldots, n) \]

where \( E(Z_n) = 0 \), \( V(Z_n) = \sigma^2 \), \( \sum b_i = 0 \), \( \sum_{t=1}^{K} n_t = N \), \( G^{(N)} = \frac{n \sum_{i=1}^{K} (\bar{x}_i - \bar{x})^2}{\sum_{t=1}^{K} \sum_{i=1}^{n_t} (x_{it} - \bar{x}_i)^2} \), \( \lambda^{(N)} = \frac{n \sum b_i^2}{\sigma^2} \) and \( \delta = \frac{\sum b_i^2}{K \sigma^2} \).

The sequential procedure, as suggested by Ray (1956), for making inference regarding the significant difference of group-means are as follows:

\[
\begin{align*}
\text{Accept } H_0 & \quad \text{if } \quad e^{\frac{1}{2} \lambda^{(N)}} \left( \frac{Kn_1 - 1}{2} \right) \left( \frac{K - 1}{2} \right) \left( \frac{1}{1 + G^{(N)}} \right) \leq \frac{\beta_2}{1 - \beta_1} \\
\text{Accept } H_1 & \quad \text{if } \quad e^{\frac{1}{2} \lambda^{(N)}} \left( \frac{Kn_1 - 1}{2} \right) \left( \frac{K - 1}{2} \right) \left( \frac{1}{1 + G^{(N)}} \right) \geq \frac{1 - \beta_2}{\beta_1} \\
\end{align*}
\]

\[ \text{Otherwise take a further set of } K \text{ observations, one from each group.} \]

In the above, \( \beta_1 \) and \( \beta_2 \) are the chances of erroneous rejection of \( H_0 \) and \( H_1 \) respectively and
\[ M(X, Y; U) = \sum_{j=0}^{\infty} \frac{\sqrt{X+jU}}{\sqrt{X+Y+j}j} \] is the confluent hyper geometric function.

Thus,

\[
M\left( \frac{Kn-1}{2}, \frac{K-1}{2}; \frac{1}{2}, \frac{1}{1+G^{(N)}} \right)
\]

\[ = \sum_{j=0}^{\infty} \frac{\sqrt{K-1}}{2} \sqrt{\frac{K}{2} + j} \left( \frac{1}{2} \frac{(nK\delta)G^{(N)}}{1+G^{(N)}} \right)^j \]

\[ = f_0 + f_1 + f_2 + \ldots \ldots \]  \hspace{1cm} \text{(2:2)}

where \( f_i = \frac{\sqrt{K-1}}{2} \sqrt{\frac{K}{2} + i} \left( \frac{1}{2} \frac{(nK\delta)G^{(N)}}{1+G^{(N)}} \right)^i \)

\[ \text{In the following section we shall fuzzify the decision procedure given in (2:1). For simplicity, let us denote the variance ratio in (2:1) by } R. \]

Thus \( R = e^{\frac{1}{2} \frac{X^{(N)}}{2} M\left( \frac{Kn-1}{2}, \frac{K-1}{2}; \frac{1}{2}, \frac{1}{1+G^{(N)}} \right)} \) \hspace{1cm} \text{(2:4)}

For the purpose of fuzzification, we consider the variable under consideration and the parameters involved to be triangular fuzzy numbers. Also assuming the probabilities of type I and type II errors to be fuzzy, we obtain the required inference rules.

3. FUZZY SEQUENTIAL DECISION PROCEDURE FOR COMPLETELY RANDOMIZED DESIGN

We define \( x_{it} \), the \( i^{th} \) observation of the \( t^{th} \) group, in terms of fuzzy triangular number as follows
Fuzzy sequential decision procedure

\[ \bar{x}_i = [x_n - a, x_n, x_n + a] \] ........................... (3:1)

The notation \( \mu_\alpha (\cdot) \) we use to denote the fuzzy membership function (f.m.f) of \( Y \). In that sense the f.m.f of \( \bar{x}_n \), denoted by \( \mu_{\bar{x}_n} (X) \) is given by

\[
\mu_{\bar{x}_n} (X) = \begin{cases} 
\frac{x - x_n - a}{a}, & x_n - a \leq x \leq x_n \\
-\frac{x - x_n + a}{a}, & x_n \leq x \leq x_n + a \\
0, & \text{otherwise}
\end{cases}
\]

Thus the interval of confidence for \( \bar{x}_n \), denoted by \( x_{n(\alpha)} \), a function of level \( \alpha (0 \leq \alpha \leq 1) \) as given by Kaufman & Gupta (1985) is –

\[ x_{n(\alpha)} = \left[ \alpha a + (x_n - a), -\alpha a + (x_n + a) \right] \] ........................... (3:3)

Also, we use the following notations -

\[ x = \text{Grand mean} = \frac{1}{nK} \sum_{i=1}^{K} \sum_{j=1}^{n} x_{ij} \]

\[ \bar{x}_i, = \text{Mean of the } i^{th} \text{ group} = \frac{1}{n} \sum_{j=1}^{n} x_{ij} ; \quad t = 1, 2, \ldots, K \]

We shall first proceed to obtain the fuzzified form of \( G^{(N)} \), defined earlier, as follows.

From (3:3) and following Talukdar and Baruah (2007), we have, the interval of confidence of the fuzzy group mean \( \bar{x}_t \), as

\[ \bar{x}_{t(\alpha)} = \left[ \alpha a + \frac{1}{n} \sum_{j=1}^{n} (x_n - a), -\alpha a + \frac{1}{n} \sum_{j=1}^{n} (x_n + a) \right] \]

Thus we have two equations to solve for level \( \alpha (0 \leq \alpha \leq 1) \), viz.,
\[ x = \alpha a + \frac{1}{n} \sum_{i=1}^{n} (x_i - a) \quad \text{and} \quad x = -\alpha a + \frac{1}{n} \sum_{i=1}^{n} (x_i + a) \]

Solving, we get
\[ \alpha = \frac{x}{a} - \frac{1}{na} \sum_{i=1}^{n} (x_i - a) \quad \text{and} \quad \alpha = -\frac{x}{a} + \frac{1}{na} \sum_{i=1}^{n} (x_i + a) \]

Therefore the f.m.f. of fuzzy \( \tilde{x}_t \) is
\[
\mu_{\tilde{x}_t}(x) = \begin{cases} \frac{x - \frac{1}{na} \sum_{i=1}^{n} (x_i - a)}{\frac{1}{n} \sum_{i=1}^{n} (x_i - a)} ; & \frac{1}{n} \sum_{i=1}^{n} (x_i - a) \leq x \leq \frac{1}{n} \sum_{i=1}^{n} (x_i + a) \\ -\frac{x}{a} + \frac{1}{na} \sum_{i=1}^{n} (x_i + a) ; & \frac{1}{n} \sum_{i=1}^{n} x_i \leq x \leq \frac{1}{n} \sum_{i=1}^{n} (x_i + a) \\ 0, & \text{otherwise} \end{cases}
\]

\[ \ldots \ldots \quad (3:4) \]

Exactly in the same way, as above, the interval of confidence of the fuzzy grand mean \( \bar{x}_t \) is
\[
\left[ \alpha a + \frac{1}{nk} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} - a), \alpha a + \frac{1}{nk} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} + a) \right]
\]

and the corresponding f.m.f. is given by
\[
\mu_{\bar{x}_t}(x) = \begin{cases} \frac{x}{a} - \frac{1}{ank} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} - a) ; & \frac{1}{nk} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} - a) \leq x \leq \frac{1}{nk} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} + a) \\ -\frac{x}{a} + \frac{1}{ank} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} + a) ; & \frac{1}{nk} \sum_{i=1}^{n} \sum_{k=1}^{k} x_{ik} \leq x \leq \frac{1}{nk} \sum_{i=1}^{n} \sum_{k=1}^{k} (x_{ik} + a) \\ 0 ; & \text{otherwise} \end{cases}
\]

\[ \ldots \ldots \quad (3:5) \]

Now, the interval of confidence of the difference of \( \tilde{x}_t \) and the fuzzy mean \( \bar{x}_t \), denoted by \((x_i - \bar{x}_t)_{(n)}\) is given by
\[
(x_i - \bar{x}_t)_{(n)} = \left\{ 2\alpha a + (x_i - a) - \frac{1}{n} \sum_{i=1}^{n} (x_i + a) \right\}
\]


\[
\left\{-2\alpha a + \left( x_i + a \right) - \frac{1}{n} \sum_{i=1}^{n} \left( x_i - a \right) \right\} \quad \text{......... (3:6)}
\]

(3:6) can extended to get the interval of confidence, \( \sum_{i} \sum_{j} (x_i - \bar{x}_j)^2(\alpha) \) as

\[
\sum_{i} \sum_{j} (x_i - \bar{x}_j)^2(\alpha) = \left[ \sum_{i} \sum_{j} \left\{ 2\alpha a + \left( x_i - a \right) - \frac{1}{n} \sum_{i} \left( x_i + a \right) \right\}^2 \right],
\]

\[
\sum_{i} \sum_{j} \left\{ -2\alpha a + \left( x_i + a \right) - \frac{1}{n} \sum_{i} \left( x_i - a \right) \right\}^2 \quad \text{......... (3:7)}
\]

Similarly, we get the following intervals of confidence -

\[
(\bar{x}_i, -\bar{x}_j) = \left[ \left\{ 2\alpha a + \frac{1}{n} \sum_{i} \left( x_i - a \right) - \frac{1}{nk} \sum_{i} \sum_{j} \left( x_i + a \right) \right\}, \right.
\]

\[
\left. \left\{ -2\alpha a + \frac{1}{n} \sum_{i} \left( x_i + a \right) - \frac{1}{nk} \sum_{i} \sum_{j} \left( x_i - a \right) \right\} \right].
\]

\[\Rightarrow \sum_{i} (\bar{x}_i, -\bar{x}_j)^2(\alpha) = \sum_{i} \left\{ 2\alpha a + \frac{1}{n} \sum_{i} \left( x_i - a \right) - \frac{1}{nk} \sum_{i} \sum_{j} \left( x_i + a \right) \right\}^2,
\]

\[
\left. \left\{ -2\alpha a + \frac{1}{n} \sum_{i} \left( x_i + a \right) - \frac{1}{nk} \sum_{i} \sum_{j} \left( x_i - a \right) \right\}^2 \right] \quad \text{......... (3:8)}
\]

From (3:7) and (3:8), the interval of confidence of the fuzzy equivalent of \( G^{(N)} \), a function of level \( \alpha (0 \leq \alpha \leq 1) \) is

\[
G^{(N)}_{(\alpha)} = \frac{n \sum_{i} (\bar{x}_i, -\bar{x}_j)^2(\alpha)}{\sum_{i} \sum_{j} (x_i - \bar{x}_j)^2(\alpha)} = \frac{n \sum_{i} \left\{ 2\alpha a + \frac{1}{n} \sum_{i} \left( x_i - a \right) - \frac{1}{nk} \sum_{i} \sum_{j} \left( x_i + a \right) \right\}^2}{\sum_{i} \sum_{j} \left\{ -2\alpha a + \left( x_i + a \right) - \frac{1}{n} \sum_{i} \left( x_i - a \right) \right\}^2},
\]
\[
\frac{n \sum_i \left( -2\alpha a + \frac{1}{n} \sum_i (x_i + a) - \frac{1}{nk} \sum_i \sum_i (x_i - a) \right)^2}{\sum_i \sum_i \left( 2\alpha a + (x_i - a) - \frac{1}{n} \sum_i (x_i + a) \right)^2}
\]

\[G_{\alpha}^N\] gives two equations to solve for level \(\alpha(0 \leq \alpha \leq 1)\) viz.

\[
x = \frac{n \sum_i \left[ 2\alpha a + \frac{1}{n} \sum_i (x_i - a) - \frac{1}{nk} \sum_i \sum_i (x_i + a) \right]^2}{\sum_i \sum_i \left[ 2\alpha a + (x_i + a) - \frac{1}{n} \sum_i (x_i - a) \right]^2}
\]

\[\text{........... (3:9)}\]

and

\[
x = \frac{n \sum_i \left[ -2\alpha a + \frac{1}{n} \sum_i (x_i + a) - \frac{1}{nk} \sum_i \sum_i (x_i - a) \right]^2}{\sum_i \sum_i \left[ 2\alpha a + (x_i - a) - \frac{1}{n} \sum_i (x_i + a) \right]^2}
\]

\[\text{........... (3:10)}\]

\[(3:9) \Rightarrow x \sum_i \sum_i \left[ 4\alpha^2 a^2 - 4\alpha a \left( x_i + a \right) - \frac{1}{n} \sum_i (x_i - a) \right] + \left( x_i + a \right) \left[ \frac{1}{n} \sum_i (x_i - a) \right] = n \sum_i \left[ 4\alpha^2 a^2 - 4\alpha a \left( \frac{1}{n} \sum_i (x_i - a) \right) + \left( \frac{1}{n} \sum_i (x_i - a) \right) \right]^2\]

After simplification we get,

\[
\alpha = \frac{p - (q + r)^{\frac{1}{2}}}{s},
\]

\[\text{.......................... (3:11)}\]

Where
Fuzzy sequential decision procedure

\[ p = \left[ x \sum_i \sum_t \left\{ \left( x_i + a \right) - \frac{1}{n} \sum_i \left( x_{i} - a \right) \right\} + \sum_t \left\{ \sum_i \left( x_i + a \right) - \frac{1}{k} \sum_t \sum_i \left( x_i + a \right) \right\} \right] \]

\[ q = \left[ x \sum_i \sum_t \left\{ \left( x_i + a \right) - \frac{1}{n} \sum_i \left( x_{i} - a \right) \right\} + \sum_t \left\{ \sum_i \left( x_i - a \right) - \frac{1}{k} \sum_t \sum_i \left( x_i + a \right) \right\} \right]^2 \]

\[ r = -nk(x-1) \left[ x \sum_i \sum_t \left\{ \left( x_i + a \right) - \frac{1}{n} \sum_i \left( x_{i} - a \right) \right\}^2 - \sum_t \left\{ \sum_i \left( x_i - a \right) - \frac{1}{k} \sum_t \sum_i \left( x_i + a \right) \right\} \right] \]

\[ s = 2nka(x-1) \]

Again

\[ (3:10) \Rightarrow x \sum_i \sum_t \left[ 4\alpha^2 a^2 + 4\alpha a \left( x_i - a \right) - \frac{1}{n} \sum_i \left( x_i + a \right) \right] + \left\{ \left( x_i - a \right) - \frac{1}{n} \sum_i \left( x_i + a \right) \right\}^2 \]

\[ = n \sum_i \left[ 4\alpha^2 a^2 - 4\alpha a \left( \frac{1}{n} \sum_i \left( x_i + a \right) - \frac{1}{nk} \sum_t \sum_i \left( x_i - a \right) \right) + \left\{ \frac{1}{n} \sum_i \left( x_i + a \right) - \frac{1}{nk} \sum_t \sum_i \left( x_i - a \right) \right\} \right]^2 \]

After simplifications we get,

\[ \alpha = \frac{-p' + \left( q' + r' \right)^{\frac{1}{2}}}{s}, \]

\[ \text{........................................... (3:12)} \]

where

\[ p' = x \sum_i \sum_t \left\{ \left( x_i - a \right) - \frac{1}{n} \sum_i \left( x_{i} + a \right) \right\} + \sum_t \left\{ \sum_i \left( x_i + a \right) - \frac{1}{k} \sum_t \sum_i \left( x_i - a \right) \right\} \]

\[ q' = \left[ x \sum_i \sum_t \left\{ \left( x_i - a \right) - \frac{1}{n} \sum_i \left( x_{i} + a \right) \right\} + \sum_t \left\{ \sum_i \left( x_i + a \right) - \frac{1}{k} \sum_t \sum_i \left( x_i - a \right) \right\} \right]^2 \]
\[ r' = -nk(x-1) \left[ x \sum_i \sum_j \left( x_{ij} - a \right) - \frac{1}{n} \sum_i \left( x_i + a \right) \right]^2 + \sum_i \left\{ \sum_j \left( x_{ij} + a \right) - \frac{1}{k} \sum_j \sum_i \left( x_{ij} - a \right) \right\}^2 \]

(3:11) and (3:12) give the f.m.f of \( G^{(N)} \) as follows

\[
\mu_{G^{(N)}}(x) = \begin{cases} 
\frac{p-(q+r)/2}{s} & \text{if } \sum_{i} \left\{ \sum_{j} \left( x_{ij} - a \right) - \frac{1}{n} \sum_{j} \left( x_{ij} + a \right) \right\}^2 \\
\frac{1}{n} \sum_{j} \left\{ \sum_{i} \left( x_{ij} + a \right) - \frac{1}{n} \sum_{i} \left( x_{ij} - a \right) \right\}^2 & \leq x \leq \frac{1}{n} \sum_{i} \left\{ \sum_{j} \left( x_{ij} - a \right) - \frac{1}{n} \sum_{i} \left( x_{ij} + a \right) \right\}^2 \\
0 & \text{otherwise} \end{cases} 
\]

Next we shall define the fuzzy forms of the other components namely, \( \beta_1, \beta_2 \) and \( \delta \), involved in the test procedure and develop the corresponding f.m.f.s. The probabilities of type I error \( \beta_1 \) and type II error \( \beta_2 \), also the parameter \( \delta \), may be defined in terms of triangular fuzzy numbers, respectively as follows
Fuzzy sequential decision procedure

\[\tilde{\beta}_1 = [\beta_1 - h, \beta_1, \beta_1 + h]\]

\[\tilde{\beta}_2 = [\beta_2 - h, \beta_2, \beta_2 + h]\]

\[\tilde{\delta} = [\delta - \epsilon, \delta, \delta + \epsilon]\]

Also ‘1’, being the total probability can be fuzzified as \(\tilde{1} = [1 - h, 1, 1]\)

Let, the fuzzy equivalents of the terms \(\frac{1 - \beta_2}{\beta_1}\) and \(\frac{\beta_2}{1 - \beta_1}\) be denoted respectively by \(A_1\) and \(A_2\)

Then from Talukdar and Baruah (2007), the interval of confidence of \(A_1\) and \(A_2\) respectively are,

\[A_{1(\alpha)} = \left[\frac{2ah + (1 - \beta_2 - 2h)}{-ah + (\beta_i + h)}, \frac{-ah + (1 - \beta_2 + h)}{ah + (\beta_i - h)}\right]\]

\[A_{2(\alpha)} = \left[\frac{\alpha h + (\beta_2 - h)}{-ah + (1 - \beta_1 + \delta)}, \frac{-ah + (\beta_2 + h)}{2ah + (1 - \beta_1 - 2h)}\right]\]

From (3:14), the two equations to be solved for level \(\alpha\) \((0 \leq \alpha \leq 1)\) are

\[x = \frac{2ah + (1 - \beta_2 - 2h)}{-ah + (\beta_i + h)}\]

\[x = \frac{-ah + (1 - \beta_2 + h)}{ah + (\beta_i - h)}\]

Solving we get,

\[\alpha = \frac{x(\beta_i + h) - (1 - \beta_2 - 2h)}{h(2 + x)}\]

\[\alpha = \frac{x(\beta_i - h) - (1 - \beta_2 + h)}{-h(1 + x)}\]

Therefore, the f.m.f. of \(A_1\) is
\[ \mu_A(x) = \begin{cases} \frac{x(\beta_1 + h) - (1 - \beta_2 - 2h)}{h(2 + x)}, & \frac{1 - \beta_2 - 2h}{\beta_1 + h} \leq x \leq \frac{1 - \beta_2}{\beta_1} \\ \frac{x(\beta_1 - h) - (1 - \beta_2 + h)}{-h(1 + x)}, & \frac{1 - \beta_2}{\beta_1} \leq x \leq \frac{1 - \beta_2 + h}{\beta_1 - h} \\ 0, & \text{otherwise} \end{cases} \]

\[ \mu_A(x) = \begin{cases} \frac{x(1 - \beta_1 + h) - (\beta_2 - h)}{h(1 + x)}, & \frac{\beta_2 - h}{1 - \beta_1 + h} \leq x \leq \frac{\beta_2}{1 - \beta_1} \\ \frac{-x(1 - \beta_1 - 2h) + (\beta_2 + h)}{-h(2x + 1)}, & \frac{\beta_2}{1 - \beta_1} \leq x \leq \frac{\beta_2 + \delta}{1 - \beta_1 - 2\delta} \\ 0, & \text{otherwise} \end{cases} \]

\[ \mu_A(x) = \begin{cases} \frac{x - (\delta - \epsilon)}{\epsilon}, & \delta - \epsilon \leq x \leq \delta \\ \frac{-x + (\delta + \epsilon)}{\epsilon}, & \delta \leq x \leq \delta + \epsilon \\ 0, & \text{otherwise} \end{cases} \]

From the f.m.fs of \( A_1 \) and \( A_2 \), it is clear that the fuzzy \( A_1 \) and fuzzy \( A_2 \) take the forms

\[ A_1 = \left[ \frac{1 - \beta_2 - 2h}{\beta_1 + h}, \frac{1 - \beta_2}{\beta_1}, \frac{1 - \beta_2 + h}{\beta_1 + h} \right] \quad \text{and} \quad A_2 = \left[ \frac{\beta_2 - h}{1 - \beta_1 + h}, \frac{\beta_2}{1 - \beta_1}, \frac{\beta_2 + h}{1 - \beta_1 - 2h} \right] \]
Fuzzy sequential decision procedure

Since \( \lambda^{(N)} = Kn\delta = N\delta \), assuming,

\[
e^{(1)} = \exp\left(-\frac{N}{2}(\delta + \epsilon)\right), \quad e^{(2)} = \exp\left(-\frac{N}{2}\delta\right) \quad \text{and} \quad e^{(3)} = \exp\left(-\frac{N}{2}(\delta - \epsilon)\right),
\]

we define the fuzzified form of \( \epsilon = \frac{1}{2}N^{(N)} \) as \( E = (e^{(1)}, e^{(2)}, e^{(3)}) \) with the corresponding f.m.f.

\[
\mu_{E}(x) = \begin{cases} 
\frac{x - e^{(1)}}{e^{(2)} - e^{(1)}} \cdot e^{(1)} \leq x \leq e^{(2)} \\
\frac{-x + e^{(3)}}{e^{(3)} - e^{(2)}} \cdot e^{(2)} \leq x \leq e^{(3)} \\
0, \text{ otherwise}
\end{cases}
\] ........................ (3:19)

Also in the same way, discussed above, the interval of confidence, a function of level of presumption \( \alpha \), \( \alpha \in [0,1] \) of \( E \) is

\[
E(\alpha) = \left[\alpha(e^{(2)} - e^{(1)}) + e^{(1)}, -\alpha(e^{(3)} - e^{(2)}) + e^{(3)}\right] = \left[\frac{N\alpha}{2} + e^{(1)}, -\frac{N\alpha}{2} + e^{(3)}\right]
\] .......... (3:20)

Next, for simplicity, we define the fuzzy \( f_i \) (given in (2:3)), denoted by \( F_i \) as

\[
F_i = \left(F_i^{(1)}, F_i^{(2)}, F_i^{(3)}\right), \quad \text{with f.m.f.}
\]

\[
\mu_{F_i}(x) = \begin{cases} 
\frac{x - F_i^{(1)}}{F_i^{(2)} - F_i^{(1)}} \cdot F_i^{(1)} \leq x \leq F_i^{(2)} \\
\frac{-x + F_i^{(3)}}{F_i^{(3)} - F_i^{(2)}} \cdot F_i^{(2)} \leq x \leq F_i^{(3)} \\
0, \text{ otherwise}
\end{cases}
\] ........................ (3:21)

(3:21) can be solved to give the interval of confidence of \( F_i \) as
\[ \left[ \alpha \left( F_i^{(2)} - F_i^{(1)} \right) + F_i^{(1)} \right] - \left[ \alpha \left( F_i^{(3)} - F_i^{(2)} \right) + F_i^{(3)} \right] \]

Therefore the interval of confidence of \( \sum_{i=0}^{N} F_i \) is

\[ \left[ \alpha \sum_{i} \left( F_i^{(2)} - F_i^{(1)} \right) + \sum_{i} F_i^{(1)} \right] - \left[ \alpha \sum_{i} \left( F_i^{(3)} - F_i^{(2)} \right) + \sum_{i} F_i^{(3)} \right] \] ............. (3:22)

(3:20) and (3:22) give the interval of confidence of \( e^{-\frac{1}{2}\lambda^{(N)}} \times \sum_{i=0}^{N} F_i \) as

\[ \left[ \left\{ \frac{N\alpha}{2} \in +e^{(1)} \right\} \left\{ \alpha \sum_{i} \left( F_i^{(2)} - F_i^{(1)} \right) + \sum_{i} F_i^{(1)} \right\}, \right. \right. \]

\[ \left. \left. \left\{ \frac{-N\alpha}{2} \in +e^{(3)} \right\} \left\{ -\alpha \sum_{i} \left( F_i^{(3)} - F_i^{(2)} \right) + \sum_{i} F_i^{(3)} \right\} \right\}, \] ............. (3:23)

Above interval gives two equation to solve for level \( \alpha(0 \leq \alpha \leq 1) \), viz.,

\[ x = \left\{ \frac{N\alpha}{2} \in +e^{(1)} \right\} \left\{ \alpha \sum_{i} \left( F_i^{(2)} - F_i^{(1)} \right) + \sum_{i} F_i^{(1)} \right\} \]

Or \[ x = \frac{N}{2} \sum_{i} \left( F_i^{(2)} - F_i^{(1)} \right) \alpha^2 + \left\{ \frac{N}{2} \sum_{i} F_i^{(1)} + e^2 \sum_{i} \left( F_i^{(2)} - F_i^{(1)} \right) \right\} \alpha + e^{(1)} \sum_{i} F_i^{(1)} \] ................. (3:24)

and

\[ x = \left\{ \frac{-N\alpha}{2} \in +e^{(3)} \right\} \left\{ -\alpha \sum_{i} \left( F_i^{(3)} - F_i^{(2)} \right) + \sum_{i} F_i^{(3)} \right\} \]

Or \[ x = \frac{N}{2} \sum_{i} \left( F_i^{(3)} - F_i^{(2)} \right) \alpha^2 - \left\{ \frac{N}{2} \sum_{i} F_i^{(3)} + e^3 \sum_{i} \left( F_i^{(3)} - F_i^{(2)} \right) \right\} \alpha + e^{(3)} \sum_{i} F_i^{(3)} \] ................. (3:25)
Solving (3:24) and (3:25), we get two roots lying in [0,1]. The roots respectively are

\[
\alpha = \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(2) - F_{i}(1) \right\} - \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(2) - F_{i}(1) \right\}^2 - 4 \left\{ \frac{N}{2} \sum_{i} F_{i}(1) - F_{i}(1) \right\} \left\{ e^{(1)} \sum_{i} F_{i}(1) \right\}
\]

and

\[
\alpha = \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)}(F_{i}(3) - F_{i}(2)) \right\} - \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(2) - F_{i}(1) \right\}^2 - 4 \left\{ \frac{N}{2} \sum_{i} F_{i}(1) - F_{i}(1) \right\} \left\{ e^{(1)} \sum_{i} F_{i}(1) \right\}
\]

Therefore, denoting the fuzzy form of the variance ratio R (defined in (2:4)) by T, finally the f.m.f. of T, the fuzzy test criterion is given

\[
\mu_{\alpha}(x) = \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(2) - F_{i}(1) \right\} - \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(2) - F_{i}(1) \right\}^2 - 4 \left\{ \frac{N}{2} \sum_{i} F_{i}(1) - F_{i}(1) \right\} \left\{ e^{(1)} \sum_{i} F_{i}(1) \right\}
\]

\[
when \quad e^{(1)} \sum_{i} F_{i}(1) \leq x \leq e^{(2)} F_{i}(2)
\]

\[
\mu_{\alpha}(x) = \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(3) - F_{i}(2) \right\} - \left\{ \frac{N}{2} \sum_{i} F_{i}(1) + e^{(1)} \sum_{i} F_{i}(3) - F_{i}(2) \right\}^2 - 4 \left\{ \frac{N}{2} \sum_{i} F_{i}(1) - F_{i}(1) \right\} \left\{ e^{(1)} \sum_{i} F_{i}(1) \right\}
\]

\[
when \quad e^{(2)} \sum_{i} F_{i}(2) \leq x \leq e^{(3)} \sum_{i} F_{i}(3)
\]

\[
0 \quad otherwise
\]

\[\text{.......................... (3:26)}\]
It is clear from the f.m.f. given in (3.26), that the fuzzy sequential test criterion for testing the difference of group means of a completely randomized design is given by

\[ T = \left( e^{(1)} \sum F_i , \ e^{(2)} \sum F_i^{(2)} , \ e^{(3)} \sum F_i^{(3)} \right) \]  

\[ \left( 3.27 \right) \]

where \( e^{(1)}, e^{(2)}, e^{(3)} \) & \( F_i \) have meanings defined earlier.

Finally, we proceed to give the required fuzzy sequential decision rules in the following section.

4. FUZZY SEQUENTIAL DECISION RULES

Comparing the values of fuzzy \( A_1 \) & fuzzy \( A_2 \) with that of the fuzzy variance ratio given in (3.27), we attain at the following fuzzy decision rules to make inference about the group means of a completely randomized design, at probability level \( \beta_1 \)

**Decision I :**

*Accept \( H_0 \) and terminate the process at nth stage if*

\[ e^{(1)} \sum F_i \leq \frac{\beta_2 - h}{1 - \beta_1 + h}, \]

\[ e^{(2)} \sum F_i^{(2)} \leq \frac{\beta_2}{1 - \beta_1} \]

and

\[ e^{(3)} \sum F_i^{(3)} \leq \frac{\beta_2 + h}{1 - \beta_1 - 2h} \]

**Decision II :**

*Accept \( H_1 \) and terminate the process at nth stage if*
\[ e^{(1)} \sum_i F_i^{(1)} \geq \frac{1 - \beta_2 - 2h}{\beta_1 + h}, \]

\[ e^{(2)} \sum_i F_i^{(2)} \geq \frac{1 - \beta_2}{\beta_1}, \]

and

\[ e^{(3)} \sum_i F_i^{(3)} \geq \frac{1 - \beta_2 + h}{\beta_1 + h}, \]

Decision III:

Continue the process, taking further observations if

\[ \frac{\beta_2 - h}{1 - \beta_1 + h} < e^{(1)} \sum_i F_i^{(1)} < \frac{1 - \beta_2 - 2h}{\beta_1 + h}, \]

\[ \frac{\beta_2}{1 - \beta_1} < e^{(2)} \sum_i F_i^{(2)} < \frac{1 - \beta_2}{\beta_1} \]

and

\[ \frac{\beta_2 + h}{1 - \beta_1 - 2h} < e^{(3)} \sum_i F_i^{(3)} < \frac{1 - \beta_2 + h}{\beta_1 + h} \]

The fuzzy decision rules given above will be discussed with the help of a numerical example, in the following section.

5. NUMERICAL EXAMPLE

As an illustration let us consider the following example from Ray W.D (1956). Suppose we require to take observations from each of the three groups and wish to test the null hypothesis \( H_0 \) that each group mean is zero against the alternative \( H_1 \) that

\[ \sigma^2 = \frac{\sum_{i=1}^{3} b_i^2}{3\sigma^2} = 0.5 \] at prescribed level of probabilities of type-I and type-II errors both equal to 0.5.
Thus \( k = \) No. of groups \( = 3; \) \( n = \) no. of observations within a group; \( \lambda = nk\delta; \) \( \delta = 0.5; \) \( \beta_1 = \beta_2 = 0.5 \)

We assume \( a = 0.1; h = 0.001 \) and \( \varepsilon = 0.005 \)

The observations along with their fuzzy forms and the fuzzy test criterion i.e. the fuzzy variance ratio (given in (3.27)) for different \( n \) are shown in the following tabulation.

<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
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<td>n</td>
<td>( x_{1i} )</td>
<td>( x_{2i} )</td>
<td>( x_{3i} )</td>
<td>( \tilde{x}_{1i} )</td>
<td>( \tilde{x}_{2i} )</td>
<td>( \tilde{x}_{3i} )</td>
</tr>
<tr>
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<td>2.3</td>
<td>-7.8</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
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<td>-4.4</td>
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<td>9.3</td>
<td>-4.5</td>
<td>-4.4</td>
<td>-4.3</td>
</tr>
<tr>
<td>3</td>
<td>15.1</td>
<td>3.8</td>
<td>13.4</td>
<td>15.5</td>
<td>15.1</td>
<td>15.2</td>
</tr>
<tr>
<td>4</td>
<td>2.8</td>
<td>5.7</td>
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<td>2.7</td>
<td>2.8</td>
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</tr>
<tr>
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<td>9.9</td>
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<tr>
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<td>2.9</td>
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<tr>
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<td>-1.1</td>
<td>2.9</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
</tbody>
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Table continued.......

<table>
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<th>(8)</th>
<th>(9)</th>
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<th>(11)</th>
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<td>( Fuzzy \ x_{1i} )</td>
<td>( Fuzzy \ x_{2i} )</td>
<td>( Fuzzy \ x_{3i} )</td>
<td>( Fuzzy \ likelihood \ ratio )</td>
</tr>
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<td>1.76</td>
<td>5.57</td>
</tr>
</tbody>
</table>

Also we have

\[
A_1 = \left( 18.607 \quad 19.0 \quad 19.408 \right)
\]

\[
A_2 = \left( 0.0515 \quad 0.052 \quad 0.0537 \right)
\]

6. CONCLUSIONS

Comparing the numerical results of the fuzzy variance ratio with that of \( A_1 \) and \( A_2 \) in section 5, it is clear that, in the example under consideration, all the three conditions given in Decision I of section 4 are satisfied at the \( 7^{th} \) stage of experimentation. Hence the process is
terminated by accepting $H_0$ at 5% level of significance at the 7th stage. Also in this particular example, we observe that both the fuzzy and non-fuzzy decision rules give the same conclusion.

Generally it is seen that although the initial considerations of the fuzzy triangular numbers are of small width, the deviations in the final fuzzy results increase considerably. But here, in the above problem we have not observed much deviations in the final fuzzy results from that of non-fuzzy results. Again in defining the triangular fuzzy number $\tilde{x}_i$, in (3:1), we have assumed the width of the fuzzy interval, ‘$h$’ to be constant for all $x_i$. But we can consider ‘$h$’ to be a variable also, whose value may depend, in some way, on $x_i$. For example, one can take different values of ‘$h$’, proportional to the corresponding values $x_i$. Attempt to such cases is avoided, as it may lead to more complications in the fuzzification procedure and in some other situation, this can be tried.

Finally we may conclude that though to attain the fuzzy decision rules, it needs a long and tedious process of mathematical computations, we can get satisfactory results.

REFERENCES


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