On the Circulant Matrices with Arithmetic Sequence

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Abstract

Firstly we have defined $C_{a,r} = (c_{ij})$ as $nxn$ matrix, where $c_{ij} \equiv a + (j - i \mod n)r$, $a$ and $r$ are real numbers. Then, we have studied the eigenvalues, determinant, spectral norm, Euclidean norm of the matrix $C_{a,r}$. Also, we have investigated the spectral norm, Euclidean norm of inverse of the matrix $C_{a,r}$.

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1 Introduction

An $nxn$ matrix $C$ is called a circulant matrix if it is of the form

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For each \( i, j = 1,2,\ldots,n \) and \( k = 0,1,2,\ldots,n-1 \), all the elements \((i,j)\) such that \( j - i \equiv k \pmod{n} \) have the same value \( c_k \); these elements form the so-called \( k \)th stripe of \( C \). Obviously, a circulant matrix is determined by its first row (or column). That is \( C = \text{cir}(c_0,c_1,\ldots,c_{n-1}) \).

The properties of circulant matrices are well known[6,3]. Let \( w \) be a primitive \( n \)th root of the unity and \( f(\lambda) = \sum_{k=0}^{n-1} c_k \lambda^k \) be the generating polynomial for the sequence \((c_k)\) (the zeroth row of the matrix \( C \)). Then, for every \( m = 0,1,2,\ldots,n-1 \), \( f(w^m) \) is an eigenvalue of the matrix \( C \) and \( x_m = (1,w^m,w^{2m},\ldots,w^{(n-1)m})^T \) is an eigenvector of \( C \) belonging to the eigenvalue \( f(w^m) \). These eigenvectors are orthogonal and, thus, the matrix \( C \) is unitarily equivalent to the diagonal matrix with diagonal entries \( f(w^m) \), \( m = 0,1,2,\ldots,n-1 \)[5]. Also, the matrix \( C \) is normal and \( \det C = \prod_{m=0}^{n-1} f(w^m) \) [3].

Let \( w \) denote \( w = e^{\frac{2\pi i}{n}} \). Then, \( w \) is a primitive \( n \)-th root of unity. Therefore, for every \( m = 0,1,2,\ldots,n-1 \), \( \lambda_m = f\left(e^{\frac{2\pi k}{n}}\right) = \sum_{k=0}^{n-1} c_k e^{\frac{2\pi mk}{n}} \) is an eigenvalue of the matrix \( C \) and \( x_m = \left(1,e^{\frac{2\pi m}{n}},e^{\frac{2\pi 2m}{n}},\ldots,e^{\frac{2\pi (n-1)m}{n}}\right)^T \) is an eigenvector of \( C \) belonging to the eigenvalue \( \lambda_m \).

derived some bounds for the spectral norm and Euclidean norm of circulant matrices with the Fibonacci and Lucas numbers.

In this paper, we have defined $n \times n$ circulant matrices $C_{a,r} = (c_{ij})$ such that $c_{ij} \equiv a + (j - i \mod n)r$, $a$ and $r$ are real numbers. Then we have studied the eigenvalues, determinant, spectral norm, Euclidean norm of the matrix $C_{a,r}$ and inverse of this matrix.

2 Main Results

Theorem 2.1. The eigenvalues of the $n \times n$ matrix $C_{a,r}$ are

$$\lambda_0 = na + \frac{n(n-1)}{2}r, \quad \lambda_m = \left(\frac{n}{e^{\frac{2\pi m}{n}} - 1}\right)^r$$

where $m = 1, 2, \ldots, n-1$.

Proof. Since $C_{a,r}$ is a circulant matrix, its eigenvalues are of the form

$$\lambda_m = \sum_{k=0}^{n-1} (a + kr) e^{\frac{2\pi m k}{n}}$$

for every $m = 0, 1, 2, \ldots, n-1$. For $m = 0$, we have

$$\lambda_0 = \sum_{k=0}^{n-1} (a + kr) = a + a + r + a + 2r + \cdots + a + (n-1)r$$

$$= na + \frac{n(n-1)}{2}r.$$

Let $x = e^{-\frac{2\pi m}{n}}$. For $m = 1, 2, \ldots, n-1$ we have

$$\lambda_m = \sum_{k=0}^{n-1} (a + kr) e^{-\frac{2\pi m k}{n}} = \sum_{k=0}^{n-1} (a + kr)x^k$$

$$= a + ax (1 + x + x^2 + \cdots + x^{n-2}) + rx (1 + 2x + 3x^2 + 4x^3 + \cdots + (n-1)x^{n-2})$$

$$= a + a \left(\frac{x - x^n}{1 - x}\right) + rx \left(\frac{d(x + x^2 + x^3 + x^4 + \cdots + x^{n-1})}{dx}\right)$$

$$= a + a \left(\frac{x - 1}{1 - x}\right) + rx \left(\frac{d(x - x^n)}{dx}\right) \left(\frac{1}{1 - x}\right)$$
\[ a_1 = a - a + rx^n \left( \frac{(1 - nx^n)(1 - x) + (x - x^n)}{(1 - x)^2} \right) \]
\[ = r \left( \frac{-nx^n + nx}{(1 - x)^2} \right) = r \left( \frac{n(x - 1)}{(1 - x)^2} \right) \]
\[ = r \frac{n}{(x - 1)} = \frac{nr}{e^n - 1}. \]

**Theorem 2.2.** The spectral norm of the \( nxn \) matrix \( C_{a,r} \) is

\[ \| C_{a,r} \|_2 = \max \left( na + \frac{n(n - 1)}{2} r, \left| \frac{nr}{2 \sin \frac{\pi}{n}} \right| \right). \]

**Proof.** Since a circulant matrix is a normal matrix, we can write

\[ \| C_{a,r} \|_2 = \max \left( \lambda_m \right) = \max \left( na + \frac{n(n - 1)}{2} r, \max \left( \lambda_m \right) \right). \]

Thus for every \( m = 1, 2, ..., n - 1 \),

\[ |\lambda_m| = \left| \frac{nr}{e^n - 1} \right| = \frac{nr}{\cos \frac{2\pi m}{n} + i \sin \frac{2\pi m}{n}} \]
\[ = \frac{|nr|}{\sqrt{\cos \frac{2\pi m}{n}^2 - 2 \cos \frac{2\pi m}{n} + 1 + \sin \frac{2\pi m}{n}^2}} \]
\[ = \frac{|nr|}{\sqrt{2 \left( 1 - \cos \frac{2\pi m}{n} \right)}} \]
\[ = \frac{|nr|}{\sqrt{2 \left( 1 - 2 \sin^2 \frac{\pi m}{n} \right)}} \]
\[ = \frac{|nr|}{\sqrt{4 \sin^2 \frac{\pi m}{n}}} = \frac{|nr|}{2 \sin \frac{\pi m}{n}}. \]

For \( m = 1 \), \( |\lambda_m| \) has maximum value. Therefore
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\[
\max_{m=1,\ldots,n-1} (|\lambda_m|) = \frac{|nr|}{2 \sin \frac{\pi}{n}}
\]

and

\[
\|C_{a,r}\|_2 = \max\left\{ na + \frac{n(n-1)}{2} r, \frac{|nr|}{2 \sin \frac{\pi}{n}} \right\}.
\]

**Theorem 2.3.** The Euclidean norm of the \(nxn\) matrix \(C_{a,r}\) is

\[
\|C_{a,r}\|_E = n \sqrt{a^2 + (n-1)ar + \frac{(n-1)(2n-1)}{6} r^2}.
\]

**Proof.** From the definition of the Euclidean norm,

\[
\|C_{a,r}\|_E^2 = n \sum_{s=0}^{n-1} (a + sr)^2 = n \sum_{s=0}^{n-1} (a + sr)^2
\]

\[
= n \left( \sum_{s=0}^{n-1} a^2 + 2ar \sum_{s=0}^{n-1} s + r^2 \sum_{s=0}^{n-1} s^2 \right)
\]

\[
= n \left( na^2 + n(n-1)ar + \frac{(n-1)n(2n-1)}{6} r^2 \right)
\]

\[
= n^2 \left( a^2 + (n-1)ar + \frac{(n-1)(2n-1)}{6} r^2 \right).
\]

Thus,

\[
\|C_{a,r}\|_E = n \sqrt{a^2 + (n-1)ar + \frac{(n-1)(2n-1)}{6} r^2}.
\]

**Theorem 2.4.** The determinant of the \(nxn\) matrix \(C_{a,r}\) is

\[
|C_{a,r}| = (-1)^{n-1} n^{n-1} r^{n-1} \left( a + \frac{n-1}{2} r \right).
\]

**Proof.** If we apply the properties of the determinant to the determinant of the matrix \(C_{a,r}\), then we have the following equalities:
If we calculate the determinant of the above matrix, we have

\[
|C_{a,r}| = \begin{vmatrix}
    a & a+r & a+2r & a+3r & \cdots & a+(n-3)r & a+(n-2)r & a+(n-1)r \\
    (n-1)r & -r & -r & -r & \cdots & -r & -r & -r \\
    (n-2)r & (n-2)r & -2r & -2r & \cdots & -2r & -2r & -2r \\
    : & : & : & : & \cdots & : & : & : \\
    3r & 3r & 3r & 3r & \cdots & (3-n)r & (3-n)r & (3-n)r \\
    2r & 2r & 2r & 2r & \cdots & 2r & (2-n)r & (2-n)r \\
    r & r & r & r & \cdots & r & r & (1-n)r
\end{vmatrix}
\]

\[
= \begin{vmatrix}
    a & r & 2r & 3r & \cdots & (n-3)r & (n-2)r & (n-1)r \\
    r & -nr & 0 & 0 & \cdots & 0 & 0 & 0 \\
    r & 0 & -nr & 0 & \cdots & 0 & 0 & 0 \\
    : & : & : & : & \cdots & : & : & : \\
    r & 0 & 0 & 0 & \cdots & -nr & 0 & 0 \\
    r & 0 & 0 & 0 & \cdots & 0 & -nr & 0 \\
    r & 0 & 0 & 0 & \cdots & 0 & 0 & -nr
\end{vmatrix}.
\]

Theorem 2.5. The adjoint of the \( n \times n \) matrix \( C_{a,r} \) is

\[
\text{Adj}(C_{a,r}) = (-1)^n \begin{pmatrix}
    n^{2} - n - 2 & n^{2} - n - 2 & \cdots & n^{2} - n - 2 \\
    n^{2} - n - 2 & n^{2} - n - 2 & \cdots & n^{2} - n - 2 \\
    : & : & \cdots & : \\
    : & : & \cdots & : \\
    n^{2} - n - 2 & n^{2} - n - 2 & \cdots & n^{2} - n - 2 \\
    n^{2} - n - 2 & n^{2} - n - 2 & \cdots & n^{2} - n - 2 \\
\end{pmatrix}
\]

\[
\text{Adj}(C_{a,r}) = (-1)^n \begin{pmatrix}
    a & a+r & a+2r & a+3r & \cdots & a+(n-3)r & a+(n-2)r & a+(n-1)r \\
    (n-1)r & -r & -r & -r & \cdots & -r & -r & -r \\
    (n-2)r & (n-2)r & -2r & -2r & \cdots & -2r & -2r & -2r \\
    : & : & : & : & \cdots & : & : & : \\
    3r & 3r & 3r & 3r & \cdots & (3-n)r & (3-n)r & (3-n)r \\
    2r & 2r & 2r & 2r & \cdots & 2r & (2-n)r & (2-n)r \\
    r & r & r & r & \cdots & r & r & (1-n)r
\end{vmatrix}
\]

\[
= (-nr)^{n-1} - r^2(-nr)^{n-2} - 2r^2(-nr)^{n-2} - \cdots - (n-1)r^2(-nr)^{n-2}
\]

\[
= (-nr)^{n-1} \left[ - \left( anr + r^2 + 2r^2 + 3r^2 + \cdots + (n-1)r^2 \right) \right]
\]

\[
= (-1)^{n-1} (n-1)^{n-2} \left[ anr + \frac{n(n-1)}{2} r^2 \right]
\]

\[
= (-1)^{n-1} n^{n-3} r^{n-1} \left[ a + \frac{(n-1)}{2} r \right].
\]
Proof. Since the adjoint of a circulant matrix is also circulant, we will calculate cofactors of the elements of the first column of the matrix $C_{a,r}$. Therefore we will obtain the elements of the first row of the matrix $\text{Adj}(C_{a,r})$.

If we calculate the cofactors of the elements $c_{jl}$ ($j=1,2,\ldots,n$) of the matrix $C_{a,r}$, we have respectively,

$$(-1)^{j} \left( -n^{2} - r - a + n^{2} - r - a \right),$$

Therefore

$$\text{Adj}(C_{a,r}) = (-1)^{c_{ir}} \begin{pmatrix}
1, 1/2, 1/2, 1/2, \ldots, 1/2, 1, 1, \ldots, 1 \end{pmatrix}$$

Theorem 2.6. The inverse of the $n\times n$ matrix $C_{a,r}$ is

$$C_{a,r}^{-1} = \frac{1}{n^{2}} \left( \begin{array}{cc}
na + n^{2} - n - 2 \frac{r}{2} & n^{2} - n + 2 \frac{r}{2} \\
na + n^{2} - n - 2 \frac{r}{2} & n^{2} - n + 2 \frac{r}{2}
\end{array} \right),$$

Proof. Since $C_{a,r}^{-1} = \frac{\text{adj}(C_{a,r})}{\det C_{a,r}}$, from theorem 2.4 and theorem 2.5, the proof is trivial.

Corollary 2.1. The eigenvalues of the $n\times n$ matrix $C_{a,r}^{-1}$ are satisfy the following equalities

$$\psi_{0} = \frac{1}{na + n(n-1) \frac{r}{2}}, \quad \psi_{m} = \frac{e^{\frac{\pi m}{n}}}{nr} - 1,$$

where $m = 1,2,\ldots,n-1$.

Theorem 2.7. The spectral norm of the $n\times n$ matrix $C_{a,r}^{-1}$ is

$$\|C_{a,r}^{-1}\|_{2} = \max \left( \begin{array}{c}
\frac{1}{na + n(n-1) \frac{r}{2}}, \quad \frac{2 \sin \left[ \frac{n}{2} \right]}{nr} \end{array} \right),$$
where $\begin{bmatrix} 1 & 1 \end{bmatrix}$ denotes exact value.

**Proof.** Since the inverse of a circulant matrix is also a normal matrix we can write

$$\|C^{-1}\|_2 = \max_{m=0,n-1} \left( |\psi_m| \right) = \max \left\{ \frac{1}{na + \frac{n(n-1)}{2}}, \max_{m=1,n-1} \left( |\psi_m| \right) \right\}.$$ 

For every $m=1,2,...,n-1$,

$$|\psi_m| = \left| \frac{e^{\frac{2\pi m}{n}} - 1}{nr} \right| = \left| \frac{\cos \frac{2\pi m}{n} + i \sin \frac{2\pi m}{n} - 1}{nr} \right|$$

$$= \frac{\sqrt{\cos^2 \frac{2\pi m}{n} - 2 \cos \frac{2\pi m}{n} + 1 + \sin^2 \frac{2\pi m}{n}}}{|nr|}$$

$$= \frac{\sqrt{2 \left( 1 - \cos \frac{2\pi m}{n} \right)}}{|nr|}$$

$$= \frac{\sqrt{2 \left( 1 - \left( 1 - 2 \sin^2 \frac{\pi m}{n} \right) \right)}}{|nr|}$$

$$= \frac{\sqrt{4 \sin^2 \frac{\pi m}{n}}}{|nr|} = \frac{2 \sin \frac{\pi m}{n}}{|nr|}.$$ 

Since $m=1,2,...,n-1$, the values $2 \sin \frac{\pi m}{n}$ and $|\psi_m|$ are maximum for $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore

$$\max_{m=1,n-1} \left( |\psi_m| \right) = \frac{2 \sin \left\lfloor \frac{n}{2} \right\rfloor}{|nr|}$$

and
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\[ \|C^{-1}_{a,r}\|_2 = \max \left\{ \frac{1}{na + \frac{n(n-1)}{2}r}, \frac{2\sin\left[\frac{n}{2}\right]}{nr} \right\}. \]

**Theorem 2.8.** The Euclidean norm of the \( nxn \) matrix \( C^{-1}_{a,r} \) is

\[ \|C^{-1}_{a,r}\|_E = \sqrt{\frac{2}{n} \left( \frac{1}{r^2} + \frac{1}{n\left(a + \frac{n-1}{2}r\right)^2} \right)}. \]

**Proof.** From the define of Euclidean norm, we have

\[
\|C^{-1}_{a,r}\|_E = \sqrt{\frac{1}{n^4 \left(a + \frac{n-1}{2}r\right)^2} \left( \left[ na + \frac{n^2 - n - 2}{2}r \right]^2 + \left[ na + \frac{n^2 - n + 2}{2}r \right]^2 \right) + 1 + 1 + \cdots + 1}
\]

\[= \frac{1}{n^3 \left(a + \frac{n-1}{2}r\right)^2} \left( \frac{2n^2a^2 + \frac{2n^4 - 4n^3 + 2n^2}{4}r^2 + 2n^3ar - 2n^2ar}{r^2} + n \right) \]

\[= \frac{1}{n^3 \left(a + \frac{n^2 - 2n + 1}{4}r^2 + anr - ar\right)} \left( \frac{2n^2\left(a^2 + \frac{n^2 - 2n + 1}{4}r^2 + nar - ar\right)}{r^2} + n \right) \]

\[= \frac{1}{n} \left( \frac{2}{r^2} + \frac{1}{n\left(a + \frac{n-1}{2}r\right)^2} \right). \]

Thus the proof is completed.

**References**


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