A Common Fixed Point Theorem for Weakly Compatible Mappings in Fuzzy-2 Metric Space

Mamta Singh and Mohit Sharma

1Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi (U.P.) India
singhmamta_dev@yahoo.com

2School of Studies in Mathematics, Vikram University Ujjain, India
mohit_sharma3231@rediffmail.com

Abstract

In this paper we prove a common fixed point theorem for four weakly compatible maps in fuzzy metric space & fuzzy-2 metric space without appeal to continuity.

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1. Introduction

by Jungck and Rhoades [5] and proved some fixed point theorems for such maps without appeal to continuity in metric space.

This paper is divided in two sections in first section a common fixed point theorem for four self mapping has been proved using the concept of weakly compatibility without appeal to continuity under suitable contractive condition. In the second section the same results have been extended for a fuzzy-2 metric space.

2. Preliminaries

**Definition 2.1** [2]. A binary operation : \([0,1] \times [0,1] \rightarrow [0,1]\) is called a continuous t-norm if \(([0,1], \ast)\) is an Abelian topological monoid with the unit 1 such that \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d, \in [0,1]\). Examples of t-norms are \(a \ast b = ab\) and \(a \ast b = \min \{a, b\}\).

**Definition 2.2** [6], [10]. The 3-tuple \((X, M, \ast)\) is called a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous t-norm and \(M\) is a fuzzy set in \(X^2 \times [0, \infty)\) satisfying the following conditions for all \(x, y, z \in X\) and \(t, s > 0\).

\[
\begin{align*}
(FM-1) \quad & M(x, y, 0) = 0 \\
(FM-2) \quad & M(x, y, t) = 1 \text{ for all } t > 0 \text{ if and only if } x = y, \\
(FM-3) \quad & M(x, y, t) = M(y, x, t), \\
(FM-4) \quad & M(x, y, t) = M(y, z, s) \leq M(x, z, t + s), \\
(FM-5) \quad & M(x, y, ) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous.} \\
(FM-6) \quad & \lim_{s \rightarrow \infty} M(x, y, t) = 1 \text{ for all } x, y \in X \text{ and } t > 0.
\end{align*}
\]

Note that \(M(x, y, t)\) can be considered as the degree of nearness between \(x\) and \(y\) with respect to \(t\). We identify \(x = y\) with \(M(x, y, t) = 1\) for all \(t > 0\). The following example shows that every metric space induces a fuzzy metric space.

**Example 2.1** Let \((X, d)\) be a metric space. Define \(a \ast b = ab\) (or \(a \ast b = \min \{a, b\}\)) for all \(x, y \in X\) and \(t > 0\),

\[
M(x, y, t) = \frac{t}{t + d(x, y)}
\]

Then, \((X, M, \ast)\) is a fuzzy metric space. It is called the fuzzy metric space induced by the metric \(d\).

**Definition 2.3.** [13] A binary operation \(* : [0,1]^3 \rightarrow [0,1]\) is called a continuous t-norm if \(([0,1], *)\) is an Abelian topological monoid with the unit 1 such that \(a_1 \ast b_1 \ast c_1 \leq a_2 \ast b_2 \ast c_2\) whenever \(a_1 \leq a_2\), \(b_1 \leq b_2\) and \(c_1 \leq c_2\) for all \(a_1, a_2, b_1, b_2, c_1, c_2 \in [0,1]\).

**Definition 2.4.** [13] The 3-tuple \((X, M, \ast)\) is called a fuzzy 2-metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous t-norm and \(M\) is a fuzzy set in \(X^3 \times [0, \infty)\) satisfying the following condition for all \(x, y, z, u \in X\) and \(t_1, t_2, t_3 > 0\).

\[
\begin{align*}
(FM'-1) \quad & M(x, y, z, 0) = 0 \\
(FM'-2) \quad & M(x, y, z, t) = 1 \text{ for all } t > 0 \text{ when at least two of the three points are equal,}
\end{align*}
\]
(FM’−3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$, 
(FM’−4) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, t_1) * M(x, u, t_2) * M(u, z, t_3)$.
This corresponds to tetrahedron inequality in 2-metric space.

(FM’−5) $M(x, y, z, t)$ : $[0, \infty) \rightarrow [0, 1]$ is left continuous.

Definition 2.5.[13] A binary operation $*: [0,1]^4 \rightarrow [0,1]$ is called a continuous $t$-norm if $([0,1], *)$ is an Abelian topological monoid with the unit 1 such that

$$a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$$

whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ and $d_1 \leq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in [0,1]$.

Lemma 2.1 [9].
(i) In a fuzzy metric space $(X, M, *)$ for all $x, y, \in X, M(x, y, .)$ is a non-decreasing function.
(ii) In a fuzzy 2-metric space $(X, M, *)$ for all $x, y, z, \in X, M(x, y, z, .)$ is a non-decreasing function.

Definition 2.7 [9]. Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, t) = 1$$

for all $t > 0$. whenever $\{x_n\}$ sequence in $X$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} x = x \in X.$$

Definition 2.8 [14]. Two maps $A$ and $B$ from a fuzzy metric space $(X, M, *)$ into itself are said to be compatible if

$$\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ sequence in $X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x \in X.$$

Definition 2.9.[12] Two self-maps $A$ and $B$ from a fuzzy metric space or a fuzzy 2-metric space $(X, M, *)$ into itself are said to be weakly compatible [8] if they commute at their coincidence points, i.e., $Ax = Bx$ implies $ABx$ and $BAx$.

Definition 2.10.[12] A pair $(A, S)$ of self-maps of a fuzzy metric space or a fuzzy 2-metric space $(X, M, *)$ is said to be semi-compatible if

$$\lim_{n \to \infty} ASx_n = Sx$$

whenever $\{x_n\}$ is sequence in $X$ such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x \in X.$$

It follows that $(A, S)$ is semi-compatible and $Ay = Sy$ then, $ASy = SAy$.

Proposition 2.1 [4]. Let $A$ and $S$ be self-maps on a fuzzy metric space $(X, M, *)$. Assume that $S$ is continuous. Then, $(A, S)$ is semi-compatible if and only if $(A, S)$ is compatible.

Definition 2.11.[12] Let $(X, M, *)$ be a fuzzy 2-metric space.
(i) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, a, t) = 1$$

for all $a \in X$ and $t > 0$.
(ii) A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if

$$\lim_{n \to \infty} M(x_{n+m}, x_n, a, t) = 1$$

for all $a \in X$, $t > 0$ and $p > 0$. 

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(iii) A fuzzy 2-metrics space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.12.**[12] Two maps A and B from a fuzzy 2-metric space \((X, M, \ast)\) in to itself are said to be compatible if

\[
\lim_{n \to \infty} M(ABx_n, BAx_n, a, t) = 1
\]

for all \(a \in X\) and \(t > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x \in X.
\]

3. Main Results :-

3.1 Theorem : Let \(A, B, S\) and \(T\) be self mappings of a complete fuzzy metric space \((X, M, \ast)\) satisfying

3.1.1 \(A(X) \subseteq T(X), \ B(X) \subseteq S(X)\)

3.1.2 \((A,S)\) and \((B,T)\) are weakly compatible pairs of mappings

3.1.3 for all \(x, y \in X, t > 0\) and \(0 < k < 1\)

3.1.4 \(M(Ax, By, k t) \geq \text{Min } \{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\}\)

\[
\lim_{t \to \infty} M(x, y, t) = 1 \quad \forall x, y \in X
\]

then \(A, B, S\) and \(T\) have a unique common fixed point

**Proof:-**

Let \(x_0 \in X\) be any arbitrary point as \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\) \(\exists x_1, x_2 \in X\), such that \(Ax_0 = Tx_1 = y_1, Bx_1 = Sx_2 = y_2\) inductively, construct sequences \(\{y_n\}\) and \(\{x_n\}\) in \(X\) such that

\[
\begin{align*}
  y_{2n+1} &= Ax_{2n} = Tx_{2n+1} \\
  y_{2n+2} &= Bx_{2n+1} = Sx_{2n+2} \\
  n &= 0, 1, 2, \ldots
\end{align*}
\]

First we show that \(\{y_n\}\) is Cauchy sequence in \(X\), by using (3.1.3) with \(X = X_{2n}\) and \(Y = X_{2n+1}\)

we have

\[
\begin{align*}
  M(Ax_{2n}, Bx_{2n+1}, kt) &= M(y_{2n+1}, y_{2n+2}, kt) \\
  &\geq \text{Min } \{M(Bx_{2n+1}, Tx_{2n+1}, t), M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\
  &\quad M(Ax_{2n}, x_{2n+1}, t)\} \\
  &= \text{Min } \{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+2}, t)\} \\
  &= \text{Min } \{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t)\} \\
  &\quad \ldots \ldots 1
\end{align*}
\]

Hence

\[
M(y_{2n+1}, y_{2n+2}, t) \geq \text{Min } \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{t}{k}\right), M\left(y_{2n}, y_{2n+1}, \frac{t}{k}\right)\right\} \quad \ldots \ldots 2
\]

By putting 2 in 1 we obtain that

\[
M(y_{2n+1}, y_{2n+2}, t)
\]
Common fixed point theorem

\[ \begin{align*}
\geq & \min \left\{ M \left( y_{2n+1}, y_{2n+2}, \frac{t}{k} \right), M \left( y_{2n}, y_{2n+1}, \frac{t}{k_1} \right), M \left( y_{2n+1}, y_{2n+1}, t \right) \right\} \\
= & \min \left\{ M \left( y_{2n+1}, y_{2n+2}, \frac{t}{k} \right), M \left( y_{2n}, y_{2n+1}, t \right) \right\} \\
\geq & \min \left\{ M \left( y_{2n+1}, y_{2n+2}, \frac{t}{k^2} \right), M \left( y_{2n}, y_{2n+1}, \frac{t}{k} \right), M \left( y_{2n+1}, y_{2n+1}, t \right) \right\} \\
= & \min \left\{ M \left( y_{2n+1}, y_{2n+2}, \frac{t}{k^2} \right), M \left( y_{2n}, y_{2n+1}, t \right) \right\} \\
\geq & \min \left\{ M \left( y_{2n+1}, y_{2n+2}, \frac{t}{k^3} \right), M \left( y_{2n}, y_{2n+1}, t \right) \right\}
\end{align*} \]

Taking \( \lim \) as \( n \to \infty \) we have
\[ M \left( y_{2n+1}, y_{2n+2}, kt \right) \geq M \left( y_{2n}, y_{2n+1}, t \right) \forall \, t > 0 \]

Similarly we can obtain that
\[ M \left( y_{2n+1}, y_{2n+2}, 2kt \right) \geq M \left( y_{2n}, y_{2n+1}, t \right) \forall \, t > 0 \]

Thus for all \( n \) and \( t > 0 \), we have
\[ M \left( y_n, y_{n+1}, kt \right) \geq M \left( y_{n-1}, y_n, t \right) \]
i.e. \( M \left( y_n, y_{n+1}, t \right) \geq M \left( y_{n-1}, y_n, t/2k \right) \geq M \left( y_{n-1}, y_n, t/k^2 \right) \geq \ldots \geq M \left( y_{n-1}, y_n, t/k^n \right) \)

Hence, \( \lim_{n \to \infty} M \left( y_n, y_{n+1}, t \right) = 1 \)

This shows that \( \{y_n\} \) is a Cauchy sequence in \( x \), which is complete, therefore \( \{y_n\} \) converges to \( z \in x \) and the subsequences
\[ \begin{align*}
\{Ax_{2n}\} & \to z, \quad \{Sx_{2n}\} \to z & \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3 \\
\{Tx_{2n+1}\} & \to z, \quad \{Bx_{2n+1}\} \to z & \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 4
\end{align*} \]

Since
\[ B(x) \subseteq S(x) \] there exists a point \( u \in x \) such that \( z = S(u) \)

Now using (3.1.3)
\[ M \left( Au, Bx_{2n+1}, a, kt \right) \geq \min \{ M \left( Bx_{2n+1}, Tx_{2n+1}, t \right), M \left( Su, Tx_{2n+1}, t \right), M \left( Au, Su, t \right), M \left( Au, Bx_{2n+1}, t \right) \}\]

Letting \( n \to \infty \) we obtain
\[ M \left( Au, z, kt \right) \geq \min \{ M \left( z, z, t \right), M \left( Su, z, t \right), M \left( Au, z, t \right), M \left( Au, z, t \right) \}\]
\[ = M \left( Au, z, t \right) \]
\[ Au = z = S(u) \]

Similarly \( A(x) \subseteq T(x) \) there exists \( v \in x \) such that \( z = TV \) using (3.1.3)
\[ M \left( Au, Bv, kt \right) \geq \min \{ M \left( Bv, Tv, t \right), M \left( Su, Tv, t \right), M \left( Au, Su, t \right), M \left( Au, Bv, t \right) \}\]
\[ = \min \{ M \left( Bv, z, t \right), M \left( z, z, t \right), M \left( z, z, t \right), M \left( z, Bv, t \right) \}\]
\[ M \left( Au, Bv, kt \right) \geq M \left( Bv, z, t \right) \]
which yields

\[ Bv = z = Tz \]

By weak compatibility of pairs

\((A,S)\) and \((B,T)\)

we have

\[ ASu = SAu \]

\[ \Rightarrow Az = Sz \]

And

\[ BTv = TBv \]

\[ \Rightarrow Bz = Tz \]

\[ M(Az, z, kt) = M(Az, Bv, kt) \]

\[ \geq \min \left\{ M(Bv, Tv, t), M(Sz, Tv, t), M(Az, Sz, t), M(Az, Bv, t) \right\} \]

\[ = \min \left\{ M(z, z, t), M(Az, z, t), M(Sz, Sz, t), M(Az, z, t) \right\} \]

\[ = M(Az, z, t) \]

Which yields

\[ Az = z = Sz \]

Again using (3) we have

\[ M(z, Bz, kt) = M(Az, Bz, kt) \]

\[ \geq \min \left\{ M(Bz, Tz, t), M(Sz, Tz, t), M(Az, Sz, t), M(Az, Bz, t) \right\} \]

\[ = \min \left\{ M(Bz, Tz, t), M(z, Tz, t), M(z, z, t), M(z, Bz, t) \right\} \]

\[ M(z, Bz, kt) = \min \left\{ M(Bz, Bz, t), M(z, Bz, t), M(z, z, t), M(z, Bz, t) \right\} \]

\[ M(z, Bz, kt) \geq M(z, Bz, t) \]

Which yields

\[ Bz = z = Tz \]

Hence

\[ Az = Bz = Sz = Tz = z \]

Now for the uniqueness of \( z \) let \( z' \) be another common fixed point of \( A, B, S, T \)

then from (3.1.3)

\[ M(z, z', kt) = M(Az, Bz', kt) \]

\[ \geq \min \left\{ M(Bz', Tz', t), M(Sz, Tz', t), M(Az, Sz, t), M(Az, Bz', t) \right\} \]

\[ = \min \left\{ M(z', z', t), M(z, z', t), M(z, z, t), M(z, z', t) \right\} \]

\[ M(z, z', t) \geq M(z, z', t) \]

Which yields

\[ z = z' \]

This completes the proof

**3.2 Theorem:** Let \( A, B, S \) and \( T \) be self mappings of a complete fuzzy 2-metric space \((X, M, *)\) satisfying

3.2.1 \( A(X) \subseteq T(X), \quad B(X) \subseteq S(X) \)

3.2.2 \((A,S)\) and \((B,T)\) are weakly compatible pairs of mappings
Common fixed point theorem

3.2.3 for all $x, y \in X$, $t > 0$ and $0 < k < 1$

$$M(Ax, By, a, k t) \geq \min \{M(By, Ty, a, t), M(Sx, Ty, a, t), M(Ax, Sx, a, t)\}$$

3.2.4 $\lim_{t \to \infty} M(x, y, z, t) = 1 \forall x, y$ and $z \in X$

then $A, B, S$ and $T$ have a unique common fixed point

**Proof:-**

Let $x_0 \in X$ be any arbitrary point as $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X) \ni x_1, x_2 \in X$, such that $Ax_0 = Tx_1 = y_1$, $Bx_1 = Sx_2 = y_2$ inductively, construct sequences $\{y_n\}$ and $\{x_n\}$ in $X$ such that

$$\begin{align*}
y'_{2n+1} &= Ax_{2n} = Tx_{2n+1} \\
y'_{2n+2} &= Bx_{2n+1} = Sx_{2n+2}
\end{align*}$$

$n = 0, 1, 2, ...$

First we show that $\{y_n\}$ is Cauchy sequence in $X$, by using (3.2.3) with $X = X_{2n}$ and $Y = X_{2n+1}$ we have

$$M(Ax_{2n}, Bx_{2n+1}, a, kt) = M(y'_{2n+1}, y'_{2n+1}, a, kt) \geq \min \{M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(Sx_{2n+1}, Tx_{2n+1}, a, t), M(Ax_{2n}, Sx_{2n}, a, t), M(Ax_{2n}, x_{2n+1}, a, t)\}$$

$$= \min \{M(y_{2n+1}, y_{2n+2}, a, t), M(y_{2n}, y_{2n+1}, a, t), M(y_{2n}, y_{2n+2}, a, t)\}$$

$$= \min \{M(y_{2n+1}, y_{2n+2}, a, t), M(y_{2n}, y_{2n+1}, a, t)\} \quad .... 1$$

Hence

$$M(y_{2n+1}, y_{2n+2}, a, t) \geq \min \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{a}{k}, t \right), M\left(y_{2n}, y_{2n+1}, \frac{a}{k}, t \right)\right\} \quad .... 2$$

By putting 2 in 1 we obtain that

$$M(y_{2n+1}, y_{2n+2}, a, t) \geq \min \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{a}{k}, t \right), M\left(y_{2n}, y_{2n+1}, \frac{a}{k}, t \right)\right\} \quad .... 3$$

$$= \min \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{a}{k}, t \right), M\left(y_{2n}, y_{2n+1}, \frac{a}{k}, t \right)\right\}$$

$$\geq \min \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{a}{k^2}, t \right), M\left(y_{2n}, y_{2n+1}, \frac{a}{k^2}, t \right)\right\} \quad .... 4$$

$$= \min \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{a}{k^2}, t \right), M\left(y_{2n}, y_{2n+1}, \frac{a}{k^2}, t \right)\right\}$$

$$\geq \min \left\{M\left(y_{2n+1}, y_{2n+2}, \frac{a}{k^m}, t \right), M\left(y_{2n}, y_{2n+1}, \frac{a}{k^m}, t \right)\right\} \quad .... 5$$

Taking lim as $n \to \infty$ we have
Similarly we can obtain that
\[ M(y_{2n+1}, y_{2n+3}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) \forall t > 0 \]
Thus for all \( n \) and \( t > 0 \), we have
\[ M(y_n, y_{n+1}, a, kt) \geq M(y_{n-1}, y_n, a, t) \]
i.e.
\[ M(y_n, y_{n+1}, a, t) \geq M\left(y_{n-1}, y_n, a, \frac{t}{k}\right) \geq M\left(y_{n-2}, y_{n-1}, a, \frac{t}{k^2}\right) \geq M\left(y_0, y_1, a, \frac{t}{k^n}\right) \]
Hence
\[ \lim_{n \to \infty} M(y_n, y_{n+1}, a, t) = 1 \]
Now, we prove by induction that for any integers \( p \)
\[ \lim_{n \to \infty} M(y_n, y_{n+p}, a, t) = 1 \]
Clearly 5 is true for \( p = 1 \), suppose that 5 is true for \( p = m \)
i.e.
\[ \lim_{n \to \infty} M(y_n, y_{n+m}, a, t) = 1 \]
Now using \((F, M, \ast)\) we have
\[ M(y_n, y_{n+m+1}, a, t) \geq M\left(y_n, y_{n+m}, a, \frac{t}{z}\right) * M\left(y_{n+m}, y_{n+m+1}, a, \frac{t}{z}\right) * M\left(y_0, y_1, a, \frac{t}{k^n}\right) \]
Therefore
\[ \lim_{n \to \infty} M(y_n, y_{n+m+1}, a, t) \geq 1 * 1 * 1 = 1 \]
hence (5) is true for \( p = m + 1 \) thus (5) holds for all \( p \) and we get that \( \{y_n\} \) is a Cauchy sequence in \( x \), which is complete, therefore \( \{y_n\} \) converges to \( z \in x \) and the subsequences
\[ \{Ax_{2n}\} \to z, \{Sx_{2n}\} \to z \]
\[ \{Tx_{2n+1}\} \to z, \{Bx_{2n+1}\} \to z \]
Since
\[ B(x) \subseteq S(x) \] there exists a point \( u \in x \) such that \( z = S(u) \)
Now using \((3.2.3)\)
\[ M(Au, Bx_{2n+1}, a, kt) \geq \min \{ M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(Su, Tx_{2n+1}, a, t), M(Au, Su, a, t), M(Au, Bx_{2n+1}, a, t) \} \]
Letting \( n \to \infty \) we obtain
\[ M(Au, z, a, kt) \geq \min \{ M(z, z, a, t), M(Su, z, a, t), M(Au, Su, a, t), M(Au, z, a t) \} = M(Au, z, a, t) \]
which implies that
\[ Au = z \] therefore
\[ Au = z = Fu \]
Similarly \( A(x) \subseteq T(x) \) there exists \( v \in x \) such that \( z = Tv \) using (3.2.3)
\[
M ( Au, Bv, a, kt ) \geq Min \{ M ( Bv, Tv, a, t ), M ( Su, Tv, a, t ), M ( Au, Su, a, t ), M ( Au, Bv, a, t ) \} \\
= Min \{ M ( Bv, z, a, t ), M ( z, z, a, t ) \} \\
M ( Au, Bv, a, kt ) = M ( Bv, z, a, t )
\]
i.e.
\[
M ( Bv, z, a, kt ) \geq M ( Bv, z, a, t )
\]
which yields
\[ Bv = z = Tv \]
By weak compatibility of pairs
\( (A, S) \) and \( (B, T) \)
we have
\[ ASu = SAu \]
\[ \Rightarrow Az = Sz \]
And
\[ BTv = TBv \]
\[ \Rightarrow Bz = Tz \]
\[
M ( Az, z, a, kt ) = M ( Az, Bv, a, kt ) \\
\geq Min \{ M ( Bv, Tv, a, t ), M ( Sz, Tv, a, t ), M ( Az, Sz, a, t ), M ( Az, Bv, a, t ) \} \\
= Min \{ M ( z, z, a, t ), M ( Az, z, a, t ), M ( Sz, Sz, a, t ), M ( Az, z, a, t ) \} \\
= M ( Az, z, a, t )
\]
which yields
\[ Az = z = Sz \]
Again using (3.2.3) we have
\[
M ( z, Bz, a, kt ) = M ( Az, Bz, a, kt ) \\
\geq Min \{ M ( Bz, Tz, a, t ), M ( Sz, Tz, a, t ), M ( Az, Sz, a, t ), M ( Az, Bz, a, t ) \} \\
= Min \{ M ( Bz, Tz, a, t ), M ( Tz, Tz, a, t ), M ( z, z, a, t ), M ( z, Bz, a, t ) \} \\
M ( z, Bz, a, kt ) = Min \{ M ( Bz, Bz, a, t ), M ( z, Bz, a, t ), M ( z, z, a, t ), M ( z, Bz, a, t ) \} \\
M ( z, Bz, a, kt ) \geq M ( z, Bz, a, t )
\]
which yields
\[ Bz = z = Tz \]
Hence
\[ Az = Bz = Sz = Tz = z \]
Now for the uniqueness of \( z \) let \( z' \) be another common fixed point of \( A, B, S, T \) then from (3.2.3)
\[
M ( z, z', a, kt ) = M ( Az, Bz', a, kt )
\]
which yields
\[ z = z' \]
This completes the proof.

For \( A = B, S = T \) in theorem 3.2 of this paper, we have the following result.

**Corollary 3.1:** Let \( A \) and \( S \) be self mappings of a complete fuzzy 2-metric space \((X, M, \ast)\) with continuous \( t \)-norm \( \ast \), defined by \( a \ast b = \min(a, b) \ a, b \in [0, 1] \) satisfying:

3.1.1 \( A(X) \subseteq S(X) \)

3.1.2 \( (A, S) \) weakly compatible pair of mappings.

3.1.3 for all \( x, y \in X, t > 0 \) and \( 0 < k < 1 \)
\[
M(Ax, Ay, a, k t) \geq \min \{M(Ay, Sy, a, t), M(Sx, a, t), M(Ax, x, a, t)\}
\]

3.1.4 \( \lim \limits_{t \to \infty} M(x, y, z, t) = 1 \ \forall \ x, y \text{ and } z \in X \)

then \( A \) and \( S \) have a unique common fixed point in \( X \).

For \( A = B, S = T = I \) in theorem 3.2 of this paper, we have the following result.

**Corollary 3.2:** Let \( A \) and \( S \) be self mappings of a complete fuzzy 2-metric space \((X, M, \ast)\) with continuous \( t \)-norm \( \ast \), defined by \( a \ast b = \min(a, b) \ a, b \in [0, 1] \) satisfying:

3.2.3 for all \( x, y \in X, t > 0 \) and \( 0 < k < 1 \)
\[
M(Ax, Ay, a, k t) \geq \min \{M(Ay, y, a, t), M(x, y, a, t), M(Ax, x, a, t)\}
\]

then \( A \) has a unique common fixed point in \( X \).

**Remark 3.1:** Results of this paper remains true even the pairs \( (A, S) \) and \( (B, T) \) are weakly commuting maps instead of weakly compatible in fuzzy 2-metric space.

**Remark 3.2:** Results of this paper remains true even the pairs \( (A, S) \) and \( (B, T) \) are commuting or weakly commuting maps and one of the maps \( S \) and \( T \) are continuous.

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