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The Radius of Univalence of Certain Analytic Functions

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Abstract

Let $f(z) = z + a_2 z^2 + ...$ be analytic and $g(z) = z + b_2 z^2 + ...$ is univalent in the unit disc $E = \{z : |z| < 1\}$ such that $\frac{f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $z \in E$. In this paper , we shall find the radius of starlikeness for the function $f(z)$ in E.

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1. Introduction

Let U denote the class of functions

$$w(z)=\sum_{k=1}^{\infty}c_k z^k \tag{1.1}$$

which are regular in the unit disc $E=\{z:|z|<1\}$ and satisfying the conditions

$$w(0)=0 \text{ and } |w(z)|<1,\ z\in E.$$

If f and g are analytic functions in E, then we say that f is subordinate to g ,written as $f\prec g$ or $f(z)<g(z)$, if there exists a function $w(z)\in U$ such that $f(z)=g(w(z))$. If g is univalent then $f<g$ if and only if $f(0)=g(0)$ and $f(E)\subset g(E)$.

Suppose that

$$f(z)=z+a_2z^2+\ldots\ldots \text{ be analytic}$$

and

$$g(z)=z+b_2z^2+\ldots\ldots \text{ is univalent in } E$$

with the conditions

$$\frac{f(z)}{g(z)}\prec\frac{1+Az}{1+Bz},\quad -1\le B<A\le 1,\quad z\in E. \tag{1.2}$$

Krzyz and Reade[3] made an early study for $A=1, B=-1$ and $A=1, B=0$ and obtained radius of starlikeness of $f(z)$. After this Goel[1] made investigations for the radius of starlikeness of $f(z)$ under the conditions $A=1, B=\frac{1}{\alpha}-1\left(\alpha>\frac{1}{2}\right)$.

We shall obtain the radius of starlikeness of $f(z)$ for $-1\le B<A\le 1$. Results due to Krzyz and Reade[3] and Goel[1] follow as special cases from our theorem.

2. Some Preliminary Lemmas

In our investigation , we shall require the following lemmas.

Lemma 2.1. If $w(z)\in U$, then for $|z|=r<1$,

$$|zw'(z)-w(z)|\le\frac{r^2-|w(z)|^2}{1-r^2} .$$

This result was obtained by Singh and Goel[4] .

Lemma 2.2. Let $p(z)=\frac{1+Bw(z)}{1+Aw(z)}$, $w(z)\in U$, then for $|z|=r<1$,

$$\operatorname{Re}\left[Ap(z)+\frac{B}{p(z)}\right]+\frac{r^2|Ap(z)-B|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}$$

$$\le\begin{cases}\dfrac{AB(A+B)r^2-4ABr+(A+B)}{(1-Ar)(1-Br)}, R_1\le R_0,\\[2ex] \dfrac{2}{(1-r^2)}\left[(1-ABr^2)-\left((1-A)(1-B)(1+Ar^2)(1+Br^2)\right)^{1/2}\right], R_1\ge R_0, A\ne 1,\end{cases}$$

where $R_1=\frac{1-Br}{1-Ar}$ and $R_0^2=\frac{(1-B)(1+Br^2)}{(1-A)(1+Ar^2)}$.

The bounds are sharp.

Goel and Mehrok [2] established this result.

Lemma 2.3. Let

$$g(z) = z + b_2 z^2 + \ldots\ldots$$

be analytic and univalent in the unit disc E. Then inequality

$$\mathrm{Re}\left[\frac{zg'(z)}{g(z)}\right] \geq \frac{1-r}{1+r} \quad \text{holds for}$$

$$|z| \leq \tanh\frac{1}{2} = 0.46212\ldots.$$

The bounds are sharp for each z.

This lemma is due to Krzyz and Reade[3] .

3. Main Result

Theorem 3.1. Let

$$f(z) = z + a_2 z^2 + \ldots\ldots$$

and

$$g(z) = z + b_2 z^2 + \ldots\ldots$$

are analytic in the unit disc E such that

$$\frac{f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \; -1 \leq B < A \leq 1, \; z \in E.$$

If $g(z)$ is univalent in E, then

(i) For $A_0 \le A \le 1$, $f(z)$ is starlike in $|z| < r_0$, where r_0 is the smallest positive root of

$$ABr^3 - B(2+A)r^2 + (1+2A)r \ -1 = 0 \ ; \tag{3.1}$$

(ii) For $-1 < A \le A_0$, $f(z)$ is starlike in $|z| < r_1$, where r_1 is the smallest positive root of

$$B(1-A)r^4 - 2B(1-A)r^3 + (1-2(A-B)-AB)r^2 - 2(1-A)r \ + (1-A) = 0; \tag{3.2}$$

$$A_0 = \left(\frac{3-\sqrt{5}}{2}\right).$$

Results are sharp .

Proof. By definition of subordination , (1.2) gives

$$\frac{f(z)}{g(z)} = \frac{1+Aw(z)}{1+Bw(z)} \ , \ w(z) \in U \ .$$

This implies that

$$f(z) = g(z)\frac{1+Aw(z)}{1+Bw(z)} \ . \tag{3.3}$$

Differentiating logarithmically , (3.3) yields

$$\left(\frac{zf'(z)}{f(z)}\right) = \frac{zg'(z)}{g(z)} + (A-B)\frac{zw'(z)}{(1+Aw(z))(1+Bw(z))} \ . \tag{3.4}$$

Taking the real parts on both sides of (3.4) and using lemma 2.1 , we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \operatorname{Re}\frac{zg'(z)}{g(z)}$$

$$+(A-B)\left[\operatorname{Re}\frac{w(z)}{(1+Aw(z))(1+Bw(z))}-\frac{r^2-|w(z)|^2}{(1-r^2)|(1+Aw(z))(1+Bw(z))|}\right]. \tag{3.5}$$

Put $p(z)=\dfrac{1+Bw(z)}{1+Aw(z)}$, $w(z)\in U$.

Then from (3.5) , we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\geq \operatorname{Re}\frac{zg'(z)}{g(z)}+\frac{(A+B)}{(A-B)}$$

$$-\frac{1}{(A-B)}\left[\operatorname{Re}\left(Ap(z)+\frac{B}{p(z)}\right)+\frac{r^2|Ap(z)-B|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\right]. \tag{3.6}$$

Since $g(z)$ is univalent , it follows from lemma 2.3 that

$$\operatorname{Re}\frac{zg'(z)}{g(z)}\geq\frac{1-r}{1+r} \ . \tag{3.7}$$

(3.6) in conjunction with (3.7) and lemma 2.2 yields

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)$$

$$\geq\begin{cases}\dfrac{1-(1+2A)r+B(2+A)r^2-ABr^3}{(1+r)(1-Ar)(1-Br)}, R_1\leq R_0,\\ -2\left[(1-A)+(A-B)r+B(1-A)r^2\right]\\ \dfrac{+2\left[(1-A)(1-B)(1+Ar^2)(1+Br^2)\right]^{1/2}}{(A-B)(1-r^2)}, R_1\geq R_0, A\neq 1.\end{cases} \tag{3.8}$$

On equating the right hand sides of (3.8) to zero , we get (3.1) and (3.2) .

The equation $R_1=R_0$ yields

$$ABr^4 - 2ABr^3 + (2A + 2B - AB - 1)r^2 - 2r + 1 = 0 \ . \tag{3.9}$$

Elimination of r between (3.1) and (3.9) leads to

$$(1+B)\left(BA^3 - 2BA^2 + 2A - 1\right) = 0. \tag{3.10}$$

If $1+B \neq 0$, we have

$$B = \frac{(2A-1)}{A^2(2-A)} \ , \ A \neq 1.$$

Then $B < A$ implies that $0 < (1-A)^3(1+A)$ which holds.

Also $B = \dfrac{(2A-1)}{A^2(2-A)} > -1$ implies $A < \dfrac{3-\sqrt{5}}{2} < 1$.

For $B = -1$, elimination of r between (3.1) and (3.9) gives

$$A^2 - 3A + 1 = 0 \ .$$

Therefore $A = \dfrac{3-\sqrt{5}}{2} = A_0$, say .

Corollary 1. By taking $A = 1$ and $B = \dfrac{1}{\alpha} - 1 \left(\alpha > \dfrac{1}{2} \right)$, $f(z)$ is univalent and starlike in $|z| < r$, where r is the smallest positive root of

$$\left(1-\frac{1}{\alpha}\right)r^3 - 3\left(1-\frac{1}{\alpha}\right)r^2 - 3\left(1-\frac{1}{\alpha}\right)r + 1 = 0.$$

This is a result proved by Goel[1] .

Corollary 2. For $A = 1$ and $B = -1$, we get the radius of starlikeness as $2-\sqrt{3}$, earlier established by Krzyz and Reade[3].

Corollary 3. Putting $A=1$ and $B=0$, $f(z)$ is univalent and starlike in $|z|<\frac{1}{3}$. This result was proved by Krzyz and Reade[3] .

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