Economical Runge-Kutta Methods with Weak Second Order for Stochastic Differential Equations

Anna Napoli

Department of Mathematics, University of Calabria
87036 Rende (Cs), Italy
a.napoli@unical.it

Abstract

Explicit economical three-stage Runge-Kutta methods for the numerical solution of stochastic differential equations are derived. They are of second order accuracy in the weak sense. Some particular methods are considered. Numerical examples are presented to support the theoretical results.

Mathematics Subject Classification: 60H10

Keywords: Stochastic differential equations, Stochastic Runge-Kutta methods, Weak approximation, Economical methods

1 Introduction

In recent years there has been much interest in designing numerical methods for stochastic differential equations (SDEs) since they are used for the description of many real-life phenomena in different fields, including biology and physics, population dynamics, economics and finance. Because of the difficulty of obtaining exact solutions to such equations, numerical methods are needed.

In this paper we consider the scalar autonomous SDE

\[ dy(t) = a(t, y(t))dt + b(t, y(t))dW_t \quad t_0 \leq t \leq T \]
\[ y(t_0) = y_0 \tag{1} \]

where \( W = \{W_t, 0 \leq t \leq T\} \) is a one-dimensional standard Wiener process, \( a \) and \( b \) are respectively the drift and the diffusion coefficient. We assume that \( a \) and \( b \) are defined and measurable in \([t_0, T] \times \mathbb{R}\) and satisfy both Lipschitz and linear growth bound conditions in \( y \). These requirements ensure existence and uniqueness of solution of the SDE (1).
Equation (1) can be written in the integral form

\[ y(t) = y_0 + \int_{t_0}^{t} a(y_s) \, ds + \int_{t_0}^{t} b(y_s) \, dW \] (2)

where the first integral is a regular Riemann-Stiltjes integral and the second one is a stochastic integral with respect to the Wiener process \( W(t) \).

Analogously with the deterministic case, by truncating the stochastic Taylor expansion for the process (2), different numerical methods can be constructed ([6]). But the computational cost can be high due to the proliferation of elementary derivatives. In order to derive derivative-free methods, the extension of classical Runge-Kutta methods to stochastic differential equations has been introduced (see [6], which includes also an extensive bibliography).

In [2] the authors gave an overview of methods of Runge-Kutta type for SDEs studied until then. In [1] and [9] new classes of stochastic Runge-Kutta schemes were derived. They are of second-order accuracy in the weak sense.

For deterministic differential equations the so-called economical Runge-Kutta methods have been proposed (see [3] and the references therein) which are less expensive then the classical Runge-Kutta methods in terms of function evaluations. In [4] and [5] the authors extended the idea of deterministic economical Runge-Kutta methods to the solution of SDEs, by constructing respectively a weak second order Runge-Kutta type method, which is the economical version of a method proposed in ([9]), and a strong global order one method.

In this paper we generalize the idea in [4] and propose explicit three-stage economical Runge-Kutta methods with weak order 2.

In order to facilitate the reading of the work, in Section 2 we quote deterministic economical Runge-Kutta methods and in Section 3 stochastic Runge-Kutta methods. In Section 4 we derive the order conditions and we find examples of families satisfying the obtained conditions. In the last Section numerical examples are given which compare the proposed method to other weak second order ones.

## 2 Deterministic economical Runge-Kutta scheme

A deterministic explicit \( s \)-stage Runge-Kutta method for the numerical solution of the differential equation \( y'(t) = f(t, y) \) with initial condition \( y(t_0) = y_0 \) is

\[
\begin{align*}
y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i K_{i,n} \quad n = 0, 1, \ldots, N - 1 \\
y_0 &= y(t_0)
\end{align*}
\] (3)
where

\[ K_{i,n} = f \left( t_n + c_i h_n, y_n + h_n \sum_{j=1}^{i-1} a_{ij} K_{j,n} \right) \]  \hspace{1cm} (4)

\[ b_i, a_{ij} \in \mathbb{R}, \ h_n = t_{n+1} - t_n, \ c_1 = 0 \text{ and, usually, } c_i = \sum_{j=1}^{i-1} a_{ij} \ i = 1, \ldots, s. \]

Equations (3)-(4) can be written in the equivalent form

\[
\begin{cases}
    Y^n_i &= y_n + h_n \sum_{j=1}^{i-1} a_{ij} f \left( t_n + c_j h_n, Y^n_j \right) \\
y_{n+1} &= y_n + h_n \sum_{i=1}^{s} b_i f \left( t_n + c_i h_n, Y^n_i \right) \hspace{1cm} n = 0, 1, \ldots, N - 1. \end{cases} \]  \hspace{1cm} (5)

A Runge-Kutta method with order \( p \geq 3 \) belongs to class \( A^p \) if \( b_1 = 0, c_s = 1 \) ([3]).

For a method of class \( A^p \), in [3] the authors proposed the economical version

\[
\begin{cases}
y_{n+1} &= y_n + h_n \sum_{i=2}^{s} b_i K_{i,n} \hspace{1cm} n = 0, 1, \ldots, N - 1 \\
y_0 &= y(t_0) \end{cases} \]  \hspace{1cm} (6)

where

\[ K_{i,n} = f \left( t_n + c_i h_n, y_n + h_n \sum_{j=2}^{i-1} a_{ij} K_{j,n} + a_{i1} K_{s,n-1} \right) \]  \hspace{1cm} (7)

In the equivalent form it is

\[
\begin{cases}
    Y^n_i &= y_n + h_n \sum_{j=2}^{i-1} a_{ij} f \left( t_n + c_j h_n, Y^n_j \right) + a_{i1} f \left( t_n + c_1 h_n, Y^n_{s-1} \right) \\
y_{n+1} &= y_n + h_n \sum_{i=2}^{s} b_i f \left( t_n + c_i h_n, Y^n_i \right) \hspace{1cm} n = 0, 1, \ldots, N - 1 \\
Y^0_{s-1} &= Y^0_1 = y_0. \end{cases} \]  \hspace{1cm} (8)

At each step one function evaluation is saved.
3 Stochastic Runge-Kutta schemes

Let’s consider an equidistant discretization \( \{t_0, \ldots, t_N\} \) of the time interval \([t_0, T]\) with stepsize \( \Delta = (T - t_0)/N \). The general form of an s-stage stochastic Runge-Kutta scheme for the solution of (1), in the case of one Wiener process, is given by

\[
\begin{align*}
Y_i &= y_n + \sum_{j=1}^{s} Z_{ij}^{(0)} a(t_n + \mu_j \Delta, Y_j) + \sum_{j=1}^{s} Z_{ij}^{(1)} b(t_n + \mu_j \Delta, Y_j) \quad i = 1, \ldots, s \\
y_{n+1} &= y_n + \sum_{j=1}^{s} z_j^{(0)} a(t_n + \mu_j \Delta, Y_j) + \sum_{j=1}^{s} z_j^{(1)} b(t_n + \mu_j \Delta, Y_j)
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
K_i^n &= a(t_n + \mu_j \Delta, y_n + \Delta \sum_{j=1}^{s} \lambda_{ij} K_j^n + \Delta W \sum_{j=1}^{s} \gamma_{ij} K_j^n) \quad i = 1, \ldots, s \\
\overline{K}_i^n &= b(t_n + \mu_j \Delta, y_n + \Delta \sum_{j=1}^{s} \lambda_{ij} K_j^n + \Delta W \sum_{j=1}^{s} \gamma_{ij} K_j^n) \quad i = 1, \ldots, s \\
y_{n+1} &= y_n + \sum_{i=2}^{s} \alpha_i K_i^n + \sum_{i=1}^{s} \beta_i \overline{K}_i^n
\end{align*}
\]

with \( \mu_1 = 0 \). \( Z^{(1)} \) and \( z^{(1)} \) are respectively arbitrary matrix and vector whose elements are random variables and \( Z^{(0)} \) and \( z^{(0)} \) are respectively the parameter matrix and vector associated with the deterministic components. If both \( Z^{(0)} \) and \( Z^{(1)} \) are strictly lower triangular, then (9) is said to be explicit, otherwise it is implicit.

The above scheme (9) or (10) can be represented in the tableau form

\[
\begin{array}{c|cc}
\mu & Z^{(0)} & Z^{(1)} \\
\hline
z^{(0)^T} & z^{(1)^T}
\end{array}
\]

A specific case is

\[
\begin{align*}
Y_i &= y_n + \Delta \sum_{j=1}^{s} \lambda_{ij} a(t_n + \mu_j \Delta, Y_j) + \Delta W \sum_{j=1}^{s} \gamma_{ij} b(t_n + \mu_j \Delta, Y_j) \quad i = 1, \ldots, s \\
y_{n+1} &= y_n + \Delta \sum_{j=1}^{s} \alpha_j a(t_n + \mu_j \Delta, Y_j) + \Delta W \sum_{j=1}^{s} \beta_j b(t_n + \mu_j \Delta, Y_j)
\end{align*}
\]

and its tableau form is

\[
\begin{array}{c|cc}
\mu & Z^{(0)} & Z^{(1)} \\
\hline
z^{(0)^T} & z^{(1)^T}
\end{array}
\]
In the area of SDEs there are two ways to measure accuracy: strong convergence and weak convergence. In the first case sample path trajectories of the numerical approximation are calculated; in the case of weak approximation only some of the moments may be of interest.

**Definition 3.1** If \( \overline{y}_N \) is the numerical approximation of \( y(t_N) \) after \( N \) steps with constant stepsize \( h = \frac{t_N - t_0}{N} \), then \( \overline{y}_N \) is said to converge weakly to \( y \) with weak order \( p \) if for each polynomial \( g \) which is \( 2(p + 1) \) times continuously differentiable there exist \( k > 0 \) (independent on \( h \)) and \( \delta > 0 \) such that for the mean value \( E(|\overline{y}_N - y(t_N)|) \) the following relation holds

\[
|E(g(\overline{y}_N)) - E(g(y_N))| \leq kh^p, \quad h \in (0, \delta).
\]

Analogously with the deterministic case, the technique for obtaining the order conditions consists in matching the truncated Runge-Kutta scheme with the stochastic Taylor series expansion of the exact solution over one step assuming exact initial values (see, for example [6]).

For a given function \( f = f(t, y) \) with \( t, y \in \mathbb{R} \) we denote \( f = f_{00} = f(t_n, y_n) \) and

\[
f_{ij} = \frac{\partial^{i+j}}{\partial t^i \partial y^j} f(t_n, y_n).
\]

With this notation, if we replace the Gaussian increments \( \Delta W \) in (9) by some simpler random variables \( \tilde{W} \) with appropriate moment properties ([6]) (for example \( N(0, \Delta) \) Gaussian random variables), the simplified order two weak Taylor scheme is given by

\[
y_{n+1} = y_n + a\Delta + b\Delta \tilde{W} + \frac{1}{2}bb_{01} \left( (\Delta \tilde{W})^2 - \Delta \right) \\
+ \frac{1}{2} (b_{10} + ab_{01} + \frac{1}{2}b^2b_{02} + ba_{01}) \Delta \Delta \tilde{W} + \frac{1}{2} (a_{10} + aa_{01} + \frac{1}{2}b^2a_{02}) \Delta^2 + R
\]

where \( R \) is the remainder term.
4 Economical stochastic Runge-Kutta methods

As in the deterministic case ([3]), if we save one function call for each step by using information from the previous step, we obtain an economical Runge-Kutta method for SDEs. Let us consider 3-stage SRK methods such that \( z_1^{(0)} = 0 \).

The following scheme

\[
\begin{aligned}
Y_i &= y_n + \Delta \sum_{j=2}^{i-1} \lambda_{ij} a(c_{nj}, Y_j) + \Delta \lambda_{i1} a(c_{nj}, Y_{n}^{\text{pre}}) + \Delta W \sum_{j=1}^{i-1} \gamma_{ij} b(c_{nj}, Y_j) \\
y_{n+1} &= y_n + \Delta \sum_{j=2}^{3} \alpha_j a(c_{nj}, Y_j) + \Delta W \sum_{j=1}^{3} \beta_j b(c_{nj}, Y_j)
\end{aligned}
\]

(13)

with \( c_{nj} = t_n + \mu_j \Delta \) is an explicit economical Runge-Kutta type method (EcSRK in what follows).

It can also be written in the form

\[
\begin{aligned}
K^n_i &= a\left(c_{nj}, y_n + \Delta \sum_{j=2}^{i-1} \lambda_{ij} K^n_j + \Delta \lambda_{i1} K^{n-1}_3 + \Delta W \sum_{j=1}^{i-1} \gamma_{ij} \overline{K}^n_j\right) \quad i = 1, \ldots, 3 \\
\overline{K}^n_i &= b\left(c_{nj}, y_n + \Delta \sum_{j=2}^{i-1} \lambda_{ij} K^n_j + \Delta \lambda_{i1} K^{n-1}_3 + \Delta W \sum_{j=1}^{i-1} \gamma_{ij} \overline{K}^n_j\right) \quad i = 1, \ldots, 3 \\
y_{n+1} &= y_n + \sum_{i=2}^{3} \alpha_i K^n_i + \sum_{i=1}^{3} \beta_i \overline{K}^n_i
\end{aligned}
\]

(14)

In order to match the truncated expansion of (13) with the simplified Taylor scheme, for the expansion of \( a \) and \( b \) we have to consider the expansion of a process \( f(t + \Delta, X_t + \Delta X) \) in terms of \( \Delta \) and \( \Delta X = X_{t+\Delta} - X_t \) ([6])

\[
f(t + \Delta, X_t + \Delta X) \overset{(2)}{\approx} f_{00} + f_{10} \Delta + f_{01} \Delta X + \left( f_{20} + b^2 f_{12} + b^3 b_{01} f_{03} + \frac{b^4}{4} f_{04} \right) \frac{\Delta^2}{2} + \left( f_{11} + \frac{b^2}{2} f_{03} \right) \Delta \Delta X + f_{02} \frac{(\Delta X)^2}{2}.
\]

(15)

The notation \( A \overset{(2)}{\approx} B \) means that replacing the variable \( A \) by \( B \) in a second-order approximation leads to an equivalent approximation.
By (15) we get that approximation (13) is 2-equivalent to

\[ y_{n+1} = y_n + (\alpha_2 + \alpha_3) a \Delta + (\beta_1 + \beta_2 + \beta_3) b \Delta \hat{W} \\
+ (\alpha_2 \mu_2 + \alpha_3 \mu_3) a_10 \Delta^2 + (\alpha_2 \lambda_{21} + \alpha_3 (\lambda_{31} + \lambda_{32})) a a_{01} \Delta^2 \\
+ \frac{1}{2} (\alpha_2 \gamma_{21}^2 + \alpha_3 (\gamma_{31} + \gamma_{32})^2) a_{02} b^2 \Delta^2 + (\beta_2 \mu_2 + \beta_3 \mu_3) b_{10} \Delta \Delta \hat{W} \\
+ (\beta_2 \lambda_{21} + \beta_3 (\lambda_{31} + \lambda_{32})) a b_{01} \Delta \Delta \hat{W} + \frac{3}{2} (\beta_2 \gamma_{21}^2 + \beta_3 (\gamma_{31} + \gamma_{32})^2) b^2 b_{02} \Delta \Delta \hat{W} \\
+ (\alpha_2 \gamma_{21} + \alpha_3 (\gamma_{31} + \gamma_{32})) a_0 b \Delta \Delta \hat{W} + 3 \beta_3 \gamma_{21} \gamma_{32} b a_{01} \Delta \Delta \hat{W} \\
+ (\beta_2 \gamma_{21} + \beta_3 (\gamma_{31} + \gamma_{32})) b b_{01} (\Delta \hat{W})^2 \\
+ (\beta_2 \lambda_{21} \gamma_{21} + \beta_3 (\lambda_{31} + \lambda_{32}) (\gamma_{31} + \gamma_{32})) a b b_{02} \Delta^2 \\
+ \frac{1}{2} \beta_3 \gamma_{21} \gamma_{32} (\gamma_{21} + 6 (\gamma_{31} + \gamma_{32})) b^2 b_{01} b_{02} \Delta^2 \\
+ \beta_3 \mu_2 \gamma_{32} b_{10} b_{01} \Delta^2 + \beta_3 \lambda_{21} \gamma_{32} a b_{01} \Delta^2 \\
+ (\gamma_{21} (\alpha_3 \gamma_{32} + \beta_3 \lambda_{32}) + (\gamma_{31} + \gamma_{32}) (\beta_2 \lambda_{21} + \beta_3 \lambda_{31})) a_0 b b_{01} \Delta^2 \\
+ (\beta_2 \mu_2 \gamma_{21} + \beta_3 \mu_3 (\gamma_{31} + \gamma_{32})) b \left( b_{11} + \frac{b^2}{2} b_{03} \right) \Delta^2 + \hat{R}.
\]

(16)

Let's now compare the Itô Taylor expansion of the exact solution with the expansion (16).

Equations (12) and (16) coincide if the coefficients satisfy the following system

\[
\begin{align*}
\alpha_2 + \alpha_3 &= 1 & \beta_1 + \beta_2 + \beta_3 &= 1 \\
\alpha_2 \mu_2 + \alpha_3 \mu_3 &= \frac{1}{2} & \alpha_2 \lambda_{21} + \alpha_3 (\lambda_{31} + \lambda_{32}) &= \frac{1}{2} \\
\alpha_2 \gamma_{21}^2 + \alpha_3 (\gamma_{31} + \gamma_{32})^2 &= \frac{1}{2} & \beta_2 \mu_2 + \beta_3 \mu_3 &= \frac{1}{2} \\
\beta_2 \lambda_{21} + \beta_3 (\lambda_{31} + \lambda_{32}) &= \frac{1}{2} & \beta_2 \gamma_{21}^2 + \beta_3 (\gamma_{31} + \gamma_{32})^2 &= \frac{1}{6} \\
\alpha_2 \gamma_{21} + \alpha_3 (\gamma_{31} + \gamma_{32}) &= \frac{1}{2} & \beta_3 \gamma_{21} \gamma_{32} &= 0 \\
\beta_2 \lambda_{21} \gamma_{21} + \beta_3 (\lambda_{31} + \lambda_{32}) (\gamma_{31} + \gamma_{32}) &= 0 & \beta_3 \gamma_{21} \gamma_{32} (\gamma_{21} + 6 (\gamma_{31} + \gamma_{32})) &= 0 \\
\beta_3 \mu_2 \gamma_{32} &= 0 & \beta_3 \lambda_{21} \gamma_{32} &= 0 \\
\beta_2 \mu_2 \gamma_{21} + \beta_3 \mu_3 (\gamma_{31} + \gamma_{32}) &= 0 & \gamma_{21} (\alpha_3 \gamma_{32} + \beta_3 \lambda_{32}) + (\gamma_{31} + \gamma_{32}) (\beta_2 \lambda_{21} + \beta_3 \lambda_{31}) &= 0
\end{align*}
\]

(17)
and $R = \frac{1}{2}bb_{01} \left[(\Delta \hat{W})^2 - \Delta \right]$.

Each solution of the above system leads to a weak order 2 schemes. Observe that in order to have a solution there must be $\alpha_2, \alpha_3, \beta_2, \beta_3 \neq 0$.

Some solutions correspond to the following tables. The proposed methods will be denoted respectively by $ERK2-1$, $ERK2-2$ and $ERK2-3$.

![Table 1: ERK2-1 method](image)

![Table 2: ERK2-2 method](image)

![Table 3: ERK2-3 method](image)

5 Numerical results

In this section numerical results from the implementation of $ERK2$ methods proposed in this paper are compared to those from the implementation of method in ([7]). The methods are implemented with constant step size on two problems for which the exact solution in terms of a Wiener process is known. In Section 4 we said that in the economical schemes at each step we use a function call of the previous step. Therefore the cost of $ERK2$ methods is less than the cost of Soheili method.

All the computations were done on a PC with Core 2 processor using Matlab 7.0 and 5000 independent simulations were generated for different stepsizes. In order to simulate the Gaussian variable $J_1$ with distribution $N(0, h)$ we used the Matlab random number generator $randn$.

The implementation determines the average error for each stepsize $h$ at the end of the interval of integration. This error and the computational work (the number of function evaluations $nfc$) for each problem are summarized in Tables 4-5.

$ERK2-1$ and $ERK2-2$ methods give results which are substantially similar each other, thus we quote the results of only one of them.
Example 5.1 Consider the SDE ([7])
\[
\begin{align*}
  dy &= -a^2 y (1 - y^2) \, dt + a (1 - y^2) \, dW \\
  y(0) &= 0
\end{align*}
\]  \tag{18}

with solution
\[y(t) = \tanh (a W(t) + \text{arctanh}(y_0)).\]

For \( a = 1.0 \) we have the following results

\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( h \) & Soheili method & ERK2 methods & ERK2-1 & ERK2-3 & nfc \\
\hline
\frac{1}{25} & 1.7316e-1 & 144000 & 6.1045e-2 & 5.6670e-2 & 120001 \\
\frac{1}{50} & 1.0971e-1 & 294000 & 5.5813e-2 & 4.0913e-2 & 245001 \\
\frac{1}{100} & 8.4066e-2 & 594000 & 5.3437e-2 & 2.7832e-2 & 495001 \\
\frac{1}{200} & 6.2554e-2 & 1194000 & 5.4159e-2 & 1.8598e-2 & 995001 \\
\frac{1}{400} & 4.6519e-2 & 2394000 & 5.6290e-2 & 1.4119e-2 & 1995001 \\
\hline
\end{tabular}

Table 4: Error and computational work in the approximation of (18).

Example 5.2 Consider the nonlinear SDE ([7])
\[
\begin{align*}
  dy &= \left( \frac{1}{3} y^{\frac{1}{3}} + 6y^2 \right) \, dt + y^2 \, dW \\
  y(0) &= 1
\end{align*}
\]  \tag{19}

The solution is
\[y(t) = \left( 2t + 1 + \frac{W(t)}{3} \right)^3\]

and the exact value of the first moment is \( E[y] = 28 \) at point \( t = 1 \). The obtained results are summarized in Table 5.

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\( \Delta \) & Soheili method & ERK2 methods & ERK2-1 & ERK2-3 & nfc \\
\hline
2^{-1} & 5.43550 & 6.138 & 30000 & 1.5922e-3 & 8.735 & 3.3116e-3 & 16.791 & 25001 \\
2^{-2} & 2.24681 & 7.719 & 90000 & 2.6121e-3 & 15.731 & 1.3068e-3 & 9.1552 & 75001 \\
2^{-3} & 5.9651e-1 & 8.741 & 210000 & 2.3298e-3 & 14.204 & 5.6734e-4 & 8.5820 & 175000 \\
2^{-5} & 1.6935e-1 & 9.926 & 930000 & 4.6786e-4 & 8.6884 & 2.5654e-4 & 8.8676 & 775000 \\
\hline
\end{tabular}

Table 5: Errors and standard deviations in the approximation of \( E[y] \) in (19).
References


Received: October, 2009