On the Inequalities for the Derivative of the Inverse Polynomial of a Complex Polynomial

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Abstract

The present article contains an attempt to get a new inequality on the derivative of the inverse polynomial of a complex polynomial.

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1 Introduction

If $P(z)$ is a polynomial of degree $n$ and $P'(z)$ is its derivative, then concerning the estimate of $|P'(z)|$ on the disk $|z| \leq 1$, we have a famous inequality due to Bernstein [5].

Theorem 1.1 If $P(z)$ is a polynomial of degree $n$, then,

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$ (1)

Equality in (1) holds for polynomials, whose all the zeros lie at the origin. So it is natural to seek improvements under appropriate assumptions on the zeros of $P(z)$. Thus Erdős conjectured and Lax [3] verified that (1) can be replaced by a better inequality, if the zeros of $P(z)$ are not in the disk $|z| < 1$.

Theorem 1.2 If $P(z)$ is a polynomial of degree $n$ with $P(z) \neq 0$ in $|z| < 1$, then,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|.$$ (2)
Equality in (2) holds for the polynomial $P(z) = z^n - 1$. Further Aziz and Dawood [1] in their paper, obtained a result concerning the minimum modulus of a polynomial $P(z)$ and an inequality on its derivative analogous to (2), as stated below.

**Theorem 1.3** If $P(z)$ is a polynomial of degree $n$ with $P(z) \neq 0$ in $|z| < 1$, then,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left[ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right].$$

(3)

2 Main result

The study on the relations between the location of the zeros of a complex polynomial and its derivative is a very fertile field for researchers and effort has been on in this direction, since more than sixty years. As a consequence, various inequalities on the complex polynomials and their derivatives have been obtained which can be found in the literature. In this context, this paper deals with the relative characteristics of polynomial and its inverse polynomial, which is a new attempt by any mathematician to explore a new avenue in this direction. Here we present an inequality relating the maximum value of a polynomial having all its zeros in $|z| \geq 1$ and the maximum value of the derivative of its inverse polynomial. This kind of approach towards the inequalities is the first one and may be useful for the further investigations in the field of geometric function theory. In fact, we prove the result in a more generalized form.

**Theorem 2.1** If $P(z)$ is a polynomial of degree $n$ with $P(z) \neq 0$ in $|z| < 1$, and $Q(z) = z^n P\left(\frac{1}{z}\right)$, then for any $\gamma$ with $|\gamma| \leq 1$,

$$\max_{|z|=1} |zQ'(z) + n \left(\frac{\gamma}{2}\right) Q(z)|$$

$$\leq \frac{n}{2} \left[ \left(|1 + \frac{\gamma}{2}| - |\frac{\gamma}{2}|\right) M + \left(|1 + \frac{\gamma}{2}| + |\frac{\gamma}{2}|\right) m \right]$$

(4)

where

$$M = \max_{|z|=1} |P(z)|, \quad m = \min_{|z|=1} |P(z)|.$$

The above inequality is best possible and equality holds for the polynomial $P(z) = \alpha + \beta z^n$ with $|\beta| \leq \left(\frac{1-|\gamma|}{1+|\gamma|}\right) |\alpha|$. 


3 Lemmas

For the proof of the theorem 2.1, we require the following lemmas. Lemma 3.1 is due to Malik and Vong [4].

Lemma 3.1 Let $Q(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \leq 1$ and $P(z)$ be a polynomial of degree not exceeding that of $Q(z)$. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for any $\gamma$ with $|\gamma| \leq 1$,

$$|zP'(z) + n \left(\frac{\gamma}{2}\right) P(z)| \leq |zQ'(z) + n \left(\frac{\gamma}{2}\right) Q(z)|$$

for $|z| = 1$.

Lemma 3.2 : If $P(z)$ is a polynomial of degree $n$ with $M = 1$, then for any $\gamma$ with $|\gamma| \leq 1$ and $|z| = 1$,

$$|zP'(z) + n \left(\frac{\gamma}{2}\right) P(z)| + |zQ'(z) + n \left(\frac{\gamma}{2}\right) Q(z)| \leq \left[1 + \frac{\gamma}{2}\right] - \left|\frac{\gamma}{2}\right|$$

where $Q(z) = z^n P\left(\frac{1}{z}\right)$.

Proof. It is obvious that for any $\alpha$ and $\beta$ with $|\alpha| = |\beta| > 1$, the polynomial $P_1(z) = \frac{P(z)}{\alpha} - \beta$ does not vanish in $|z| \leq 1$. The polynomial $Q_1(z) = z^n P_1\left(\frac{1}{z}\right) = \frac{Q(z)}{\beta} - \beta z^n$ has all its zeros in $|z| \leq 1$ and hence the polynomial $Q_2(z) = \frac{Q(z)}{\beta} - \alpha z^n$ has all its zeros in $|z| \leq 1$. Also $|P_1(z)| = |Q_2(z)|$ for $|z| = 1$. Hence by lemma 3.1 we have, for $\gamma$ with $|\gamma| \leq 1$,

$$|zP_1'(z) + n \left(\frac{\gamma}{2}\right) P_1(z)| \leq |zQ_2'(z) + n \left(\frac{\gamma}{2}\right) Q_2(z)|.$$

$$\Rightarrow \left|\frac{zP'(z) + n \left(\frac{\gamma}{2}\right) P(z)}{\alpha} - n\beta \left(\frac{\gamma}{2}\right)\right| \leq \left|\frac{zQ'(z) + n \left(\frac{\gamma}{2}\right) Q(z)}{\beta} - n\alpha z^n \left(1 + \frac{\gamma}{2}\right)\right|.$$

$$\Rightarrow |zP'(z) + n \left(\frac{\gamma}{2}\right) P(z) - n\alpha \beta \left(\frac{\gamma}{2}\right)|$$

$$\leq \left|\frac{\alpha(zQ'(z) + n \left(\frac{\gamma}{2}\right) Q(z))}{\beta} - n|\alpha|^2 z^n \left(1 + \frac{\gamma}{2}\right)\right|.$$

Without loss of generality, we can choose the arguments of $\alpha$, $\beta$, and $\frac{\gamma}{2}$, such that,

$$|zP'(z) + n \left(\frac{\gamma}{2}\right) P(z)| + n |\alpha \beta \left(\frac{\gamma}{2}\right)|$$

$$\leq \left|\frac{\alpha(zQ'(z) + n \left(\frac{\gamma}{2}\right) Q(z))}{\beta} - n|\alpha|^2 z^n \left(1 + \frac{\gamma}{2}\right)\right|. \quad (5)$$
Applying lemma 3.1 to the polynomials \( \frac{\alpha}{\beta} Q(z) \) and \( |\alpha|^2 z^n \), we get,
\[
\left| \frac{\alpha(zQ'(z) + n \left( \frac{\gamma}{2} \right) Q(z))}{\beta} \right| \leq n|\alpha|^2 \left| 1 + \frac{\gamma}{2} \right|
\]
for \( |z| = 1 \). Hence the inequality (5) becomes,
\[
\left| zP'(z) + n \left( \frac{\gamma}{2} \right) P(z) \right| + n \left| \alpha \beta \left( \frac{\gamma}{2} \right) \right| \leq n|\alpha|^2 \left| 1 + \frac{\gamma}{2} \right| - \left| \frac{\alpha(zQ'(z) + n \left( \frac{\gamma}{2} \right) Q(z))}{\beta} \right|.
\]
As \( |\alpha| \to 1 \) and \( |\beta| \to 1 \), we get the desired result.

Lemma 3.3 : If \( P(z) \) is a polynomial of degree \( n \) with all its zeros in \( |z| \leq 1 \), then for any \( \gamma \) with \( |\gamma| \leq 1 \) and \( |z| = 1 \),
\[
|zP'(z) + n \left( \frac{\gamma}{2} \right) P(z)| \geq mn \left| 1 + \frac{\gamma}{2} \right|.
\]
Proof: Applying lemma 3.1 to the polynomials \( P(z) \) and \( m z^n \), we can easily get the above inequality.

4 Proof of the Theorem 2.1

If \( P(z) \) has a zero on \( |z| = 1 \), then the theorem can be proved directly by lemma 3.1 and lemma 3.2. Therefore assume that \( P(z) \) has all its zeros in \( |z| > 1 \). Now for any \( \alpha \) and \( \beta \) with \( 0 < |\alpha| = |\beta| < 1 \), we have, \( |\alpha \beta| m < m \leq |P(z)| \) for \( |z| = 1 \). Hence by Rouche's theorem, the polynomial \( P(z) - \alpha \beta m \) has no zeros in \( |z| < 1 \). But then \( p_1(z) = \frac{P(z)}{\alpha} - \beta m \) does not vanish in \( |z| < 1 \). Hence the polynomial \( Q_1(z) = z^n p_1(\frac{1}{z}) = \frac{Q(z)}{\alpha} - \beta z^n m \) has all its zeros in \( |z| \leq 1 \), by which we can conclude that the polynomial \( Q_2(z) = \frac{Q(z)}{\beta} - m \alpha z^n \) has all its zeros in \( |z| \leq 1 \). Also \( |P_1(z)| = |Q_2(z)| \) for \( |z| = 1 \). Hence by lemma 3.1, we get,
\[
|zP_1'(z) + n \left( \frac{\gamma}{2} \right) P_1(z)| \leq |zQ_2'(z) + n \left( \frac{\gamma}{2} \right) Q_2(z)|
\]
for \( |z| = 1 \), by which it follows that,
\[
\left| \frac{zP'(z) + n \left( \frac{\gamma}{2} \right) P(z)}{\alpha} - mn\beta \left( \frac{\gamma}{2} \right) \right| \leq \left| \frac{zQ'(z) + n \left( \frac{\gamma}{2} \right) Q(z)}{\beta} - mn\alpha z^n \left( 1 + \frac{\gamma}{2} \right) \right|
\]
\[
\Rightarrow |zP'(z) + n \left( \frac{\gamma}{2} \right) P(z) - mn\alpha \beta \left( \frac{\gamma}{2} \right)|
\]
\[
\leq \left| \alpha(zQ'(z) + n \left( \frac{\gamma}{2} \right) Q(z)) - mn|\alpha|^2 z^n \left( 1 + \frac{\gamma}{2} \right) \right|.
\]
Now we can choose the arguments of $\alpha$, $\beta$, and $\frac{\alpha}{\beta}$, such that

$$|zP'(z) + n\left(\frac{\alpha}{\beta}\right) P(z)| + mn |\alpha \beta \left(\frac{\gamma}{\beta}\right)|$$

$$\leq \left| \frac{\alpha(zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z))}{\beta} \right| - mn|\alpha|^2 |z|^n \left| 1 + \frac{\gamma}{2} \right|.$$  \hspace{1cm} (6)

Since the polynomial $Q(z)$ has all its zeros in $|z| \leq 1$ and

$$\min_{|z|=1} |P(z)| = m = \min_{|z|=1} |Q(z)|$$

by lemma 3.3 we have,

$$mn \left| 1 + \frac{\gamma}{2} \right| \leq \left| zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z) \right|, \quad |\gamma| \leq 1, \quad |z| = 1.$$

Hence the inequality (6) becomes,

$$|zP'(z) + n\left(\frac{\alpha}{\beta}\right) P(z)| + mn |\alpha \beta \left(\frac{\gamma}{\beta}\right)|$$

$$\leq \left| \frac{\alpha(zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z))}{\beta} \right| - mn|\alpha|^2 \left| 1 + \frac{\gamma}{2} \right|.$$  \hspace{1cm} (6)

As $|\alpha| \to 1$, $|\beta| \to 1$, we get,

$$\left| zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z) \right| - \left| zP'(z) + n\left(\frac{\alpha}{\beta}\right) P(z) \right|$$

$$\leq mn \left( \left| 1 + \frac{\gamma}{2} \right| + \left| \frac{\gamma}{2} \right| \right), \quad |z| = 1. \hspace{1cm} (7)$$

But by lemma 3.2 we have,

$$\left| zP'(z) + n\left(\frac{\alpha}{\beta}\right) P(z) \right| + \left| zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z) \right|$$

$$\leq Mn \left( \left| 1 + \frac{\gamma}{2} \right| - \left| \frac{\gamma}{2} \right| \right), \quad |z| = 1. \hspace{1cm} (8)$$

Addition of (7) and (8) gives the desired inequality. Hence the proof is complete.

References


[4] Malik and Vong, Inequalities concerning the derivative of polynomials, 

[5] Schaeffer A.C., Inequalities of a Markoff and S Bernstein for polynomials 

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