Study of an Expanding, Spherical Gas Bubble in a Liquid under Gravity when the Centre Moves in a Vertical Plane

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Abstract

An expanding spherical gas bubble moves in an incompressible inviscid fluid under gravity, acting vertically downward, in such a way that the centre of the bubble moves in a vertical plane. The non-linear equations of motion are obtained and solved numerically for different values of the various parameters of the problem. The path traced by the centre of the bubble and velocity of the centre, the change of radius $R$ with time, and the influence of the buoyancy force, which is experienced by the expanding bubble for different values of the gravitational acceleration on these quantities, are investigated. The gravitational acceleration is found to influence the results significantly.

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1 Introduction

Rayleigh [1] obtained the basic differential equation giving the growth of radius \( R(t) \) of a spherical gas bubble with time when the bubble expanded adiabatically in an incompressible inviscid fluid. This equation was modified by Plesset [2] by taking into account the viscosity of the liquid.

Rayleigh’s formula has been modified when the translation of the bubble through the surrounding inviscid liquid is considered and the deviation of the bubble shape from its spherical form is also taken into account in [3]. The last study has been modified [4] when the liquid through which the bubble moves is taken to be viscous but the bubble shape is restricted to be spherical in form. The detailed study of the motion of the bubble considered in [4] has been made through numerical computation in [5].

In all the above studies the presence of gravitational acceleration has been ignored, and the bubble centre has been considered to move along a straight line in references [3], [4] and [5]. Chakraborty and Tuteja [6] have investigated the influence of gravitational acceleration on the results in bubble dynamics obtained in references [1], [2], [4] and [5]. They found that gravitational acceleration influences the bubble motion significantly.

Longuet-Higgins, Kerman and Lunde [7] considered the problem of release of an air bubble from an underwater nozzle directed vertically downward along which gravitational acceleration acts. We shall here consider the motion of the bubble when it is released from the nozzle in a direction making an angle with the vertical. The bubble centre will not now move along a vertical line, but will move in a vertical plane. We shall find the path and velocity of the bubble centre and the growth of radius of the bubble as time progresses. The gravitational acceleration is found to influence the results significantly. The bubble has an oscillatory motion in the beginning but, as the height of the bubble above a fixed horizontal plane increases, this oscillatory motion vanishes.

In the present paper we shall first obtain three simultaneous non-linear differential equations which describe the motion of a spherical, expanding gas bubble through an incompressible, inviscid liquid in the presence of a gravitational acceleration acting vertically downward, the centre of the bubble moving in a vertical plane. The height of the centre of the bubble above a fixed horizontal plane is found to increase monotonically as time progresses. The radius \( R(t) \) of the bubble as well as the horizontal and vertical velocity components \( U_y \), \( U_z \), respectively, of the bubble centre are found to vary periodically with time when the acceleration due to gravity is small. But when the acceleration due to gravity increases, this periodicity in the values of \( R(t) \), \( U_y \) and \( U_z \) with \( t \) is lost.

The path traced by the centre of the bubble as time progresses is obtained
by solving the differential equations of the problem. The case when \( \dot{R} = 0 \) [non-expanding bubble] and when \( \dot{R} \neq 0 \) [expanding bubble] are compared. The effect of buoyancy is felt more when \( \dot{R} \neq 0 \), since for an expanding bubble the upward thrust felt due to buoyancy is more, as the volume of the displaced liquid is more, the radius \( R \) of the expanding bubble being large than that for a non-expanding bubble [\( \dot{R} = 0 \)]. The acceleration of the bubble centre as well as its velocity in the vertical direction are, therefore, expected to be larger for an expanding bubble [\( \dot{R} \neq 0 \)] than for a bubble of constant radius [\( \dot{R} = 0 \)]. The path of the bubble centre in the former case is expected to rise vertically earlier than in the latter case and results of numerical computations confirm this expectation.

2 Formulation of the problem

An adiabatically expanding spherical gas bubble of radius \( R[t] \) is projected in a liquid of density \( \rho \) with a velocity \( \mathbf{U} = (0, U_y, U_z) \) from a nozzle in the presence of a gravitational acceleration \( \mathbf{g} = (0, 0, -g) \) which acts vertically along the \( z \)-axis. We shall use the Lagrangian procedure [8] to obtain the basic equations of motion. As shown in [6], the kinetic energy \( T \) and the potential energy \( V \) can be given as

\[
T = \pi \rho R^3 (2\dot{R}^2 + U^2/3),
\]

\[
V = \frac{4}{3} \pi R^3 \left[ P_e(t) + \frac{P_g}{\gamma - 1} \left( \frac{R_0}{R} \right)^{3\gamma} + \frac{3\sigma}{R} \right] - \frac{4}{3} \pi R^3 \rho gz.
\]

The expression for \( V \), as given in (2), can be obtained by the following consideration. In the absence of gravitational acceleration \( (g = 0) \), the potential energy \( V \) is given by [5].

\[
V = \frac{4}{3} \pi R^3 \left[ P_e(t) + \frac{P_g}{\gamma - 1} \left( \frac{R_0}{R} \right)^{3\gamma} + \frac{3\sigma}{R} \right].
\]

Here \( \rho \) and \( \sigma \) are the density and the surface tension of the liquid and \( P_e(t) \) is the pressure in the liquid at a large distance from the bubble, \( \gamma \) is the ratio of the two specific heats of the gas which is expanding adiabatically in the bubble. A dot represents differentiation with time. In the presence of the gravitational acceleration, the potential energy of the liquid in the spherical region of radius \( R \), with centre at a height \( z \) above a horizontal plane, is \( \frac{4}{3} \pi R^3 \rho gz \). When this liquid is removed and a bubble radius \( R \) is created, then the potential energy \( V \) of the system is reduced by the above amount and hence the expression for \( V \) will be given by (2).
The velocity potential function $\phi$ is given (see [4]) by

$$\phi = UR^3 \cos \theta/2r^2 + R^2 \dot{R}/r$$

(3)

in spherical polar co-ordinates, with the axis of symmetry being taken along the direction of the velocity $U$. The centre is taken as origin.

We define the Lagrangian $L$ as

$$L = T - V$$

(4)

and take $R, y$ and $z$ as independent co-ordinates, where $(0, y, z)$ are the co-ordinates of the centre of the spherical bubble, the $z$-axis is along the vertically upward direction and $g = (0, 0, -g)$ is along the opposite direction. The bubble is assumed to be projected initially in $y$-$z$ plane. It remain subsequently in the same plane. We also note that

$$U_y = \dot{y}$$

(5)

and

$$U_z = \dot{z}$$

(6)

We use the Lagrangian equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0,$$

(7)

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = 0$$

(8)

And

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{R}} \right) - \frac{\partial L}{\partial R} = 0$$

(9)

The equations (7)-(9) finally give us the following equations

$$R\ddot{R} + \frac{3\dot{R}^2}{2} - \frac{U^2}{4} + \frac{1}{\rho} \left\{ P_e(t) - P_g0 \left( \frac{R_0}{R} \right)^{3\gamma} + \frac{2\sigma}{R} \right\} - gz = 0,$$

(10)

$$3\dot{R}U_y + R\dot{U}_y = 0$$

(11)

and

$$3\dot{R}U_z + R\dot{U}_z - 2gR = 0$$

(12)
3 Dimensionless forms of the equations

We take \( R_0, U_0 \) and \( \rho_0 \) as the characteristic length, speed and density, respectively. We have already taken \( R_0 \) to be the initial bubble radius, \( \rho_0 \) can be taken as the density of the liquid through which the bubble rises.

We take \( R_0/U_0 \) and \( \rho_0 U_0^2 \) as characteristic time and pressure, respectively. We can then define the dimensionless quantities \( y', z', R', t', \rho', p', p_g \) and \( \bar{g} \) as:

\[
    y' = y/R_0, \quad z' = z/R_0, \quad R' = R/R_0, \quad t' = t/(R_0/U_0), \quad U'_y = U_y/U_0, \quad U'_z = U_z/U_0, \quad \rho' = \rho/\rho_0, \quad p' = p/(\rho_0 U_0^2), \quad p_g = p_g/(\rho_0 U_0^2), \quad \bar{g} = g/(U_0^2/R_0).
\]

We define the dimensionless number \( W \) [Weber number] as:

\[
    W = \frac{U_0^2 R_0 \rho_0}{\sigma}.
\]

Dividing both side of (10) by \( U_0^2 \), we can write this equation in dimensionless form as

\[
    R'd^2R'/dt'^2 + \frac{3}{2} \left( \frac{dR'/dt'}{R'} \right)^2 - \frac{1}{4} U'^2 + \frac{1}{\rho} \left\{ p' - p_g(R')^{-3\gamma} + 2/(R'W) \right\} - \bar{g} z' = 0 \quad (10')
\]

Similarly dividing the equations (11) and (12) by \( R_0 U_0^2 \) we obtain

\[
    3 \frac{dR'}{dt'} U'_y + R' \frac{dU'_y}{dt'} = 0 \quad (11')
\]

And

\[
    3 \frac{dR'}{dt'} U'_z + R' \frac{dU'_z}{dt'} - 2\bar{g} R' = 0 \quad (12')
\]

Dropping the dashes from the dimensionless quantities and using the same symbol for the dimensionless quantities as for the dimensional ones [except in the case of \( g \), gravitational acceleration] we write the last three equations (10'), (11') and (12') as

\[
    R \frac{d^2R}{dt^2} + \frac{3}{2} \left( \frac{dR}{dt} \right)^2 - \frac{1}{4} U^2 + \frac{1}{\rho} \left\{ p - p_g(R)^{-3\gamma} + 2/(RW) \right\} - \bar{g} z = 0 \quad (13)
\]

\[
    3 \frac{dR}{dt} U_y + R \frac{dU_y}{dt} = 0 \quad (14)
\]

and

\[
    3 \frac{dR}{dt} U_z + R \frac{dU_z}{dt} - 2\bar{g} R = 0 \quad (15)
\]

We also note that \( \frac{dy}{dt} = U_y, \frac{dz}{dt} = U_z \).
4 Numerical discussion

Further discussion of the bubble dynamics is based on the numerical solution of the equations (13)-(15) for various parameters of the problem. We give below some features of the bubble dynamics, as revealed through numerical computations.

(a) Path of the bubble centre. The gas bubble will experience an upward thrust due to buoyancy effect which will vary with time as the bubble expands during its motion. The bubble motion and the path in the $y$-$z$ plane traced by the bubble centre will thus be influenced by the buoyancy effect. This buoyancy effect experienced by the bubble will be larger when it expands [and hence its value increases with time] than when it has a fixed radius. We have made computations for the path of the centre of the bubble in two cases, when radius of the bubble $R$ is fixed ($\dot{R} = 0$) and also when this radius changes with time ($\dot{R} \neq 0$).

In the case when bubble expands ($\dot{R} \neq 0$) the path of the centre of the bubble in the $y$-$z$ plane is obtained solving (13)-(15) numerically but when the bubble radius is fixed ($\dot{R} = 0$) this path can be given analytically.

Equations (14) and (15) when $\dot{R} = 0$ reduce to

$$\frac{dU_y}{dt} = 0$$  \hspace{1cm} (16)

and

$$\frac{dU_z}{dt} - 2\bar{g} = 0$$  \hspace{1cm} (17)

On integrating, in view of the initial conditions $y = z = 0$, $U_y = U_{y0}$, $U_z = U_{z0}$ at $t = 0$, we obtain

$$y = U_{y0}t$$  \hspace{1cm} (18)

and

$$z = \bar{g}t^2 + U_{z0}t$$  \hspace{1cm} (19)

Eliminating $t$ from (18) and (19) we obtain

$$z = \frac{y}{U_{y0}}^2 + \left(\frac{y}{U_{y0}}\right)U_{z0}$$  \hspace{1cm} (20)

which is a parabola passing through $y = 0$, $z = 0$.

Taking $U_{y0} = 7.07$, $U_{z0} = -7.07$, as a representative case, the paths are obtained from (20) by taking $\bar{g} = 0.01, 0.05, 0.1, 1.0$. The results are shown
in Figure 1. Similar results were obtained for other values of $U_y$, $U_z$ and $\bar{g}$. As $\bar{g}$ increases, the upward buoyancy thrust on the bubble increases and the path takes an upward turn earlier in $y$-$z$ plane, $z$-axis being directed upward.

![Figure 1](image-url)

**Figure 1**
Paths of the centre of the bubble when $\dot{R} = 0$
at $\bar{g} = .01, .05, .1$ and 1

When $\dot{R} \neq 0$, the equations (13)-(15) are solved simultaneously numerically for $R$, $y$ and $z$ by Runge-Kutta method. Figure 2 and 3 show the paths when $U_y = 7.07$, $U_z = -7.07$ but $\bar{g} = .01, .05$ and $\bar{g} = 0.1, 1$ respectively. It is evident from the figures that as $\bar{g}$ increases, the path due to increasing buoyancy effect takes a sharper upward turn in the $y$-$z$ plane.
We may compare the paths in $\dot{R} = 0$ and $\dot{R} \neq 0$ cases, when the parameters $\bar{g}$, $U_{y0}$, $U_{z0}$ of the problem remain unchanged. We find that because of increased buoyancy effect when $\dot{R} \neq 0$, the paths of the bubble centre takes sharper vertically upward turns much earlier.

The numerical results for other values of physical parameters $U_{y0}$, $U_{z0}$ are similar.

(b) Variation of $R$, $U_y$ and $U_z$ with time $t$. The equation (13)-(15) are solved numerically for different values of the parameters $p_e$, $p_{g0}$, $W$ and $\bar{g}$, to obtain $R$, $U_y$, $U_z$, $y$ and $z$ as time $t$ progresses.

In Figure 4, we give the result of computations for $\bar{g} = .01$, $U_{y0} = 7.07$, $U_{z0} = -7.07$, $R(0) = 1$ and $\dot{R}(0) = 0$. The bubble is thus projected with a velocity $U(0) = 10$ downward making an angle $135^\circ$ with the upward vertical direction [z-axis].

The graph shows a periodicity in the variations of $R$, $U_y$ and $U_z$ with time for the low value .01 of $\bar{g}$. $z$ is found to increase monotonically with time $t$. 
Study of an expanding, spherical gas bubble

Figure 4

Variations of radius $R$, velocity components $U_y$ and $U_z$ and height $z$ of the centre of the bubble with time $t$ at $\bar{g} = .01$

The calculations are repeated with the above values of the parameters but with $\bar{g}$ taken as .05, 0.1 and 1. The periodicity in the variations of $R$, $U_y$ and $U_z$ with time $t$, observed when $\bar{g} = .01$, is found to be lost (see Figures 5, 6 and 7).

Figure 5

Variations of radius $R$, velocity components $U_y$ and $U_z$ and height $z$ of the centre of the bubble with time $t$ at $\bar{g} = .05$

Figure 6

Variations of radius $R$, velocity components $U_y$ and $U_z$ and height $z$ of the centre of the bubble with time $t$ at $\bar{g} = 0.1$
5 Concluding remarks

We have discussed the motion of an adiabatically expanding gas bubble in a liquid acted upon by gravitational acceleration acting vertically downward when the bubble shape is taken as spherical. The bubble is initially projected downward making an angle with the vertical downward direction. As the bubble expands it experiences a buoyancy force which increases with time. The path of the centre of the bubble lies in a vertical plane and it turns upward earlier in the case when $\dot{R} \neq 0$ since buoyancy force experienced by the bubble as it expands is more in the case than when $\dot{R} = 0$.

The bubble motion is oscillatory when $g$ is small and during the earlier stage of motion when $z$ is small.

References

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Study of an expanding, spherical gas bubble


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