On k-Regularity of Block Fuzzy Matrices

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Abstract

In this paper, necessary and sufficient conditions are given for the k-regularity of block fuzzy matrices in terms of the schur complements of its k-regular diagonal blocks. A formula for k-g-inverse of a block fuzzy matrix is established. A set of conditions for a block matrix to be expressed as the sum of k-regular block matrices is obtained.

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1 Introduction

Let $\mathcal{F}$ be a fuzzy algebra over the support $[0,1]$ with max-min operations $(+,\cdot)$ defined as $a+b=\max\{a,b\}$ and $a\cdot b=\min\{a,b\}$ for all $a,b\in[0,1]$ and the standard order ‘$\leq$’ of real numbers. Let $\mathcal{F}_{mn}$ be the set of all $m\times n$ fuzzy matrices over $\mathcal{F}$. In short $\mathcal{F}_n$ denotes $\mathcal{F}_{nn}$. For $A\in\mathcal{F}_n$, $A^T$, $R(A)$ and $C(A)$ denote the transpose, row space and column space of $A$ respectively. $A\in\mathcal{F}_{mn}$ is said to be regular if there exists $X$ such that $AXA=A$. In this case $X$ is called g-inverse of $A$. In [1] a criteria for a fuzzy matrix to be regular is given. Recently, Meenakshi and Jenita [4] have extended the notion of regular matrices to k-regular matrices for some positive integer $k$.

In this paper, we investigate the k-regularity of block fuzzy matrices of the form:
M = \[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
where A, B, C, D are fuzzy matrices of appropriate dimensions.

In section 2, some basic definitions and results are given. In section 3, equivalent conditions for k-regularity of block fuzzy matrix are determined. The structure of k-g-inverse of a block fuzzy matrix to be expressed as the sum of k-regular block matrices is obtained.

2. Preliminaries

In this section, we shall present definitions and results required from our earlier work.

Definition 2.1[4]:
A matrix $A \in \mathbb{F}_n$, is said to be right k-regular if there exists a matrix $X \in \mathbb{F}_n$ such that $A^kXA = A^k$, for some positive integer $k$. $X$ is called a right k-g-inverse of $A$. Let $A_r \{1^k\} = \{X/ A^kXA = A^k \}$.

Definition 2.2[4]:
A matrix $A \in \mathbb{F}_n$, is said to be left k-regular if there exists a matrix $Y \in \mathbb{F}_n$ such that $YA^k = A^k$, for some positive integer $k$. $Y$ is called a left k-g-inverse of $A$. Let $A_l \{1^k\} = \{Y/ AYA^k = A^k \}$.

In general, right k-g-inverse and left k-g-inverse of a fuzzy matrix are distinct [3].

Definition 2.3[2]:
Let $A = (a_{ij})$ and $B = (b_{ij})$, then $A \geq B \iff a_{ij} \geq b_{ij}$ for all $i,j \iff A+B = A$.

Lemma 2.1[1]:
For $A, B \in \mathbb{F}_n$, $R(B) \subseteq R(A) \iff B=XA$ for some $X \in \mathbb{F}_n$, $C(B) \subseteq C(A) \iff B=AY$ for some $Y \in \mathbb{F}_n$.

Lemma 2.2[4]:
For $A, B \in \mathbb{F}_n$ and a positive integer $k$, the following hold.
(i) If $A$ is right k-regular and $R(B) \subseteq R(A^k)$ then $B=BXA$ for each right k-g-inverse $X$ of $A$.
(ii) If $A$ is left k-regular and $C(B) \subseteq C(A^k)$ then $B=AYB$ for each left k-g-inverse $Y$ of $A$.

Theorem 2.1[4]:
Let $A \in \mathbb{F}_n$ and $k$ be a positive integer. The following statements are equivalent:
(i) $A$ is k-regular
(ii) $\lambda A$ is k-regular for $\lambda \neq 0 \in \mathbb{F}$.
(iii) $PAPT$ is k-regular for some permutation matrix $P$.

Theorem 2.2[5]:
Let $A \in \mathcal{F}_n$ be a $k$-regular fuzzy matrix, $C \in \mathcal{F}_{mn}$ and $B \in \mathcal{F}_{np}$, if $R(C) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$ then $CXB$ is invariant for all choices of $k$-g-inverses $X$ of $A$.

**Theorem 2.3** [3]:

Let $M$ be of the form
\[
\begin{bmatrix}
A & 0 \\
C & D
\end{bmatrix}
\]
with $A$ is right $k_1$-regular and $D$ is right $k_2$-regular. If $R(C) \subseteq R(A^k)$ and $C(C) \subseteq C(D)$, then $M$ is right $k$-regular where $k=\max\{k_1, k_2\}$.

**Theorem 2.4** [3]:

Let $M$ be of the form
\[
\begin{bmatrix}
A & 0 \\
C & D
\end{bmatrix}
\]
with $A$ is left $k_1$-regular and $D$ is left $k_2$-regular. If $R(C) \subseteq R(A)$ and $C(C) \subseteq C(D^k)$, then $M$ is left $k$-regular where $k=\max\{k_1, k_2\}$.

### 3. $k$-Regular Block Fuzzy Matrices:

In this section, we shall derive equivalent conditions for $k$-regularity of a block fuzzy matrix of the form

\[(3.1) \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]
with the diagonal blocks $A$ and $D$ are $k$-regular.

With respect to this partitioning a schur complement of $A$ in $M$ is a matrix of the form $M/A = D - CXB$, where $X$ is some $k$-g-inverse of $A$. similarly $M/D = A - BYC$ is a schur complement of $D$ in $M$.

In Theorem (2.2), it is shown that under certain conditions $CXB$ is invariant for all choices of $k$-g-inverses $X$ of $A$. By $M/A$ is a fuzzy matrix, we mean that $CXB$ is invariant and $D \supseteq CXB$. By using Definition (2.3), we note that

\[(3.2) \quad M/A \text{ is a fuzzy matrix } \iff CXB \text{ is invariant and } D = D + CXB.
\]

**Lemma 3.1:**

For $A,B,C \in \mathcal{F}_{mn}$, the following statements hold:

(i) If $R(C) \subseteq R(A^k)$, then $A$ is right $k$-regular $\iff \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right $k$-regular.

(ii) If $C(C) \subseteq C(A^k)$, then $A$ is left $k$-regular $\iff \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is left $k$-regular.

**Proof:**

Let $M = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$, then it can be easily verified that $M^k = \begin{bmatrix} A^k & 0 \\ C A^{k-1} & 0 \end{bmatrix}$.

From Lemma (2.2), if $R(C) \subseteq R(A^k)$ and $A$ is right $k$-regular then $C = CXA$ $\quad \quad (3.3)$ for each right $k$-g-inverse $X$ of $A$.

We claim that $m = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ is a right $k$-g-inverse of $M$.

\[M^k m M = \begin{bmatrix} A^k & 0 \\ C A^{k-1} & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^k & 0 \\ C A^{k-1} & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.\]
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\[
\begin{bmatrix}
A^kX & 0 \\
CA^{k-1}X & 0
\end{bmatrix}
\]

Since \(A\) is right \(k\)-regular by Definition (2.1), \(A^kX = A^k\). From (3.3), \(C = CXA\).

Therefore, \(M^kM = \begin{bmatrix}
A^kX & 0 \\
CA^{k-1}X & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A^kX & 0 \\
CXAA^{k-1}X & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A^kX & 0 \\
CXAX & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A^k & 0 \\
CXAX & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A^k & 0 \\
CA^{k-1} & 0
\end{bmatrix}
\]

\(= M^k\)

Hence \(M\) is right \(k\)-regular.

Conversely, \(M = \begin{bmatrix}
A & 0 \\
C & 0
\end{bmatrix}\) is right \(k\)-regular and by Lemma (2.1), \(R(C) \subseteq R(A^k) \Rightarrow C = XA^k\) for some \(X \in \mathcal{F}_n\).

Hence \(M = \begin{bmatrix}
A & 0 \\
C & 0
\end{bmatrix}\)

\[
= \begin{bmatrix}
A & 0 \\
X & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I & 0 \\
X & 0
\end{bmatrix} \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\]

\(= UA'\)

Where \(U = \begin{bmatrix}
I & 0 \\
X & 0
\end{bmatrix}\) and \(A' = \begin{bmatrix}
A & 0
0 & 0
\end{bmatrix}\).

Since \(M\) is right \(k\)-regular \(\Rightarrow UA'\) is right \(k\)-regular.

Hence \((UA')^k M' UA' = (UA')^k\) where \(M'\) is a right \(k\)-g-inverse of \(UA'\).

Since \((UA')^k = U(A')^k\), \((UA')^k M' UA = U(A')^k \quad \text{(3.4)}\).

For \(U' = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}\), \(U'U = 1\) and \(U'\) is a right \(k\)-g-inverse of \(U\), premultiplying (3.4) by \(U'\),

\(UU(A')^k M' UA' = U'U(A')^k \Rightarrow (A')^k (M'U) A' = (A')^k\). Thus \(A'\) is right \(k\)-regular.

Hence \(A\) is right \(k\)-regular.
(ii) Can be proved in the same manner.

**Lemma 3.2:**

For $A, B \in F_{mn}$, the following statements hold:

(i) $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is $k$-regular $\iff$ $\begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$ is $k$-regular.

(ii) $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is $k$-regular $\iff$ $\begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}$ is $k$-regular.

**Proof:**

From Theorem (2.1), $M$ is $k$-regular $\iff$ $PMP^T$ is $k$-regular for some permutation matrix $P$.

(i) $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is $k$-regular $\iff$ $P \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} P^T$ is $k$-regular for some permutation matrix $P$

$\iff$ $\begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$ is $k$-regular

(ii) Can be proved in the same manner.

**Theorem 3.1:**

Let $M$ be of the form (3.1).

(i) If either $R(C) \subseteq R(A^k)$ and $R(B) \subseteq R(D^k)$ (or) $C(B) \subseteq C(A^k)$ and $C(C) \subseteq C(D^k)$, then $M$ can be expressed as sum of two $k$-regular block matrices.

(ii) If either $R(C) \subseteq R(A^k)$, $C(C) \subseteq C(D)$, $C(B) \subseteq C(A)$ and $R(B) \subseteq R(D^k)$, then $M$ can be expressed as sum of two $k$-regular triangular block matrices.

**Proof:**

From (3.1), $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with the diagonal blocks $A$ and $D$ are $k$-regular.

(i) Since $A$ is $k$-regular and $R(C) \subseteq R(A^k)$ by Lemma(3.1), $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right $k$-regular.

Since $D$ is $k$-regular and $R(B) \subseteq R(D^k)$ by Lemma(3.1), $\begin{bmatrix} D & 0 \\ B & 0 \end{bmatrix}$ is right $k$-regular.

Hence by Lemma (3.2), $\begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}$ is right $k$-regular.

Thus $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}$ is sum of two right $k$-regular block matrices.

Similarly, if $C(B) \subseteq C(A^k)$ and $C(C) \subseteq C(D^k)$ then $M = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}$ is sum of two left $k$-regular block matrices. Thus (i) holds.
(ii) From Theorem (2.3), if \(A\) and \(D\) are \(k\)-regular with the conditions \(R(C) \subseteq R(A^k)\) and \(C(C) \subseteq C(D)\) then \([A \ 0 \ C \ D]\) is \(k\)-regular.

From Theorem (2.4), if \(A\) and \(D\) are \(k\)-regular with the conditions \(C(B) \subseteq C(A)\) and \(R(B) \subseteq R(D^k)\) then \([0 \ B \ C \ D]\) is \(k\)-regular.

Hence \(M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}\) is sum of two \(k\)-regular triangular block matrices. Thus (ii) holds.

**Theorem 3.2:**

Let \(M\) be of the form (3.1) with \(R(C) \subseteq R(A^k)\), \(C(C) \subseteq C(D^k)\), \(C(B) \subseteq C(A^k)\) and \(R(B) \subseteq R(D^k)\), the schur complements \(M/A\) and \(M/D\) are fuzzy matrices, then \(M\) is \(k\)-regular and \(m = \begin{bmatrix} X + XBYC & XBY \\ YCX & Y \end{bmatrix}\) is a \(k\)-g- inverse of \(M\) for some \(k\)-g inverses \(X\) of \(A\) and \(Y\) of \(D\) respectively.

**Proof:**

Since \(A\) is \(k\)-regular with \(R(C) \subseteq R(A^k)\) and \(C(B) \subseteq C(A^k)\) by Lemma (2.2), \(C = CXA, B = AXB\) for each \(k\)-g-inverse \(X\) of \(A\).

Since \(M/A\) is a fuzzy matrix by (3.2), it follows that \(CXB\) is invariant for all choices of \(k\)-g-inverses \(X\) of \(A\) and \(D = D + CXB\).

Now, under the given conditions, \(M\) can be expressed as \(M = ULV\) where

\[
U = \begin{bmatrix} I & 0 \\ CX & I \end{bmatrix}, \quad L = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} I & XB \\ 0 & I \end{bmatrix}.
\]

Let us define \(L^- = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\) where \(X\) is a \(k\)-g-inverse of \(A\) and \(Y\) is a \(k\)-g-inverse of \(D\).

On computation, we see that \(VL^-U = V\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}U = m\) defined in (3.5).

By using induction on \(k\), let us prove that \(M\) is \(k\)-regular.

For \(k = 1\), the result is precisely Theorem (3.1) of [2].

For \(k = 2\),

\[
M^2mM = (ULV)^2(VL^-U)(ULV).
\]

Since \(U\) and \(V\) are idempotent matrices, \(M^2mM = (ULV)^2L^-L(UV) = M^2L^-M\).

Let \(M^2 = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}\) then \(M^2 = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}\).

Hence \(P = A^2 + BC\) and \(Q = AB + BD\) \hspace{1cm} (3.6).

\[
M^2mM = M^2L^-M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]
Now, we claim that $M^2 m M = M^2 L M = M^2$.

First we prove that the $11^{th}$ block of $M^2 L M$ and that of $M^2$ are equal. For this, it is enough to prove that $A^2 + BC = PXA + QYC$.

By induction hypothesis, the given conditions reduce to the following:

- $A$ is 2-regular $\Rightarrow A^2 XA = A^2$ by Definition (2.1) (3.7).
- $C(C) \subseteq C(D^2) \Rightarrow C = DYC$ by Lemma (2.2) (3.8).
- $R(C) \subseteq R(A^2) \Rightarrow C = CXA$ by Lemma (2.2) (3.9).

By (3.2), $M/D$ is a fuzzy matrix $\Rightarrow A = A + BYC$ (3.10).

By using equations (3.6) to (3.10) yields,

- $PXA + QYC = A^2 XA + BCXA + ABYC + BDYC$
- $= A^2 + BC + ABYC + BC$
- $= A^2 + ABYC + BC$
- $= A(A + BYC) + BC$
- $= A.A + BC$
- $= A^2 + BC$

Thus, the $11^{th}$ block of $M^2 L M$ and the $11^{th}$ block of $M^2$ are equal.

Similarly, it can be verified that the remaining blocks of $M^2 L M$ and $M^2$ are equal.

Hence $M$ is 2-regular.

Assume that $M^{k-1} L^* M = M^{k-1}$, then

$M^k L^* M = M(M^{k-1} L^* M) = MM^{k-1} = M^k$.

Hence $M^k m M = M^k L^* M = M^k$. Thus $M$ is $k$-regular and $m$ is a $k$-g-inverse of $M$. Hence the theorem.

References


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