Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 24, 1169 - 1176

On k-Regularity of Block Fuzzy Matrices

A. R. Meenakshi and P. Jenita *

Department of Mathematics
Karpagam University, Coimbatore-641021, India

* Karpagam college of Engineering
Coimbatore - 641 032, India
sureshjenita@yahoo.co.in

Abstract

In this paper, necessary and sufficient conditions are given for the k-regularity of block fuzzy matrices in terms of the schur complements of its k-regular diagonal blocks. A formula for k-g-inverse of a block fuzzy matrix is established. A set of conditions for a block matrix to be expressed as the sum of k-regular block matrices is obtained.

Mathematics Subject Classification: 15B15; 15A09

Keywords: Block fuzzy matrices, k-regular Fuzzy matrices, schur complements

1 Introduction

Let $\mathcal{F}$ be a fuzzy algebra over the support [0,1] with max-min operations $(+,\cdot)$ defined as $a+b=\max\{a,b\}$ and $a\cdot b=\min\{a,b\}$ for all $a,b\in[0,1]$ and the standard order '$\leq$' of real numbers. Let $\mathcal{F}_{mn}$ be the set of all $m\times n$ fuzzy matrices over $\mathcal{F}$. In short $\mathcal{F}_n$ denotes $\mathcal{F}_{nn}$ .For $A\in \mathcal{F}_n$, A^T, R(A) and C(A) denote the transpose, row space and column space of A respectively. $A\in\mathcal{F}_{mn}$ is said to be regular if there exists X such that AXA=A. In this case X is called g- inverse of A. In [1] a criteria for a fuzzy matrix to be regular is given. Recently, Meenakshi and Jenita [4] have extended the notion of regular matrices to k-regular matrices for some positive integer k.

In this paper, we investigate the k-regularity of block fuzzy matrices of the form:

M=$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are fuzzy matrices of appropriate dimensions.

In section 2, some basic definitions and results are given. In section 3, equivalent conditions for k-regularity of block fuzzy matrix are determined. The structure of k-g-inverse of a block fuzzy matrix to be expressed as the sum of k-regular block matrices is obtained.

2. Preliminaries

In this section, we shall present definitions and results required from our earlier work.

Definition 2.1[4]:

A matrix $A\in\mathcal{F}_n$, is said to be right k-regular if there exists a matrix $X\in\mathcal{F}_n$ such that $A^kXA=A^k$, for some positive integer k. X is called a right k-g-inverse of A. Let $A_r\{1^k\}=\{X/\ A^kXA=A^k\}$.

Definition 2.2[4]:

A matrix $A\in\mathcal{F}_n$, is said to be left k-regular if there exists a matrix $Y\in\mathcal{F}_n$ such that $AYA^k=A^k$, for some positive integer k. Y is called a left k-g-inverse of A. Let $A_\ell\{1^k\}=\{Y/\ AYA^k=A^k\}$.

In general, right k-g-inverse and left k-g-inverse of a fuzzy matrix are distinct [3].

Definition 2.3[2]:

Let $A=(a_{ij})$ and $B=(b_{ij})$, then $A\geq B \Leftrightarrow a_{ij}\geq b_{ij}$ for all i,j $\Leftrightarrow$ A+B=A.

Lemma 2.1[1]:

For $A,B\in\mathcal{F}_n$, $R(B)\subseteq R(A) \Leftrightarrow B=XA$ for some $X\in\mathcal{F}_n$, $C(B)\subseteq C(A) \Leftrightarrow B=AY$ for some $Y\in\mathcal{F}_n$.

Lemma 2.2[4]:

For $A, B\in\mathcal{F}_n$ and a positive integer k, the following hold.

(i) If A is right k-regular and $R(B)\subseteq R(A^k)$ then B=BXA for each right k-g-inverse X of A .

(ii) If A is left k-regular and $C(B)\subseteq C(A^k)$ then B=AYB for each left k-g-inverse Y of A .

Theorem 2.1[4]:

Let $A\in\mathcal{F}_n$ and k be a positive integer. The following statements are equivalent:

(i) A is k-regular

(ii) λA is k-regular for $\lambda\neq 0\in\mathcal{F}$.

(iii)PAP^T is k-regular for some permutation matrix P.

Theorem 2.2[5]:

Let $A \in \mathcal{F}_n$ be a k-regular fuzzy matrix, $C \in \mathcal{F}_{mn}$ and $B \in \mathcal{F}_{np}$, if $R(C) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$ then CXB is invariant for all choices of k-g-inverses X of A.

Theorem 2.3[3]:

Let M be of the form $\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ with A is right k_1-regular and D is right k_2-regular. If $R(C) \subseteq R(A^k)$ and $C(C) \subseteq C(D)$, then M is right k-regular where $k=\max\{k_1,k_2\}$.

Theorem 2.4[3]:

Let M be of the form $\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ with A is left k_1-regular and D is left k_2-regular. If $R(C) \subseteq R(A)$ and $C(C) \subseteq C(D^k)$, then M is left k-regular where $k=\max\{k_1,k_2\}$.

3. k-Regular Block Fuzzy Matrices:

In this section, we shall derive equivalent conditions for k-regularity of a block fuzzy matrix of the form

(3.1) $M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with the diagonal blocks A and D are k-regular.

With respect to this partitioning a schur complement of A in M is a matrix of the form M/A=D-CXB, where X is some k-g-inverse of A. similarly M/D=A-BYC is a schur complement of D in M.

In Theorem (2.2), it is shown that under certain conditions CXB is invariant for all choices of k-g-inverses X of A. By M/A is a fuzzy matrix, we mean that CXB is invariant and $D \geq CXB$. By using Definition (2.3), we note that

(3.2) M/A is a fuzzy matrix $\Leftrightarrow$ CXB is invariant and D=D+CXB.

Lemma 3.1:

For $A,B,C \in \mathcal{F}_{mn}$, the following statements hold:

(i) If $R(C) \subseteq R(A^k)$, then A is right k-regular $\Leftrightarrow \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right k-regular.

(ii) If $C(B) \subseteq C(A^k)$, then A is left k-regular $\Leftrightarrow \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is left k-regular.

Proof:

Let $M=\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$, then it can be easily verified that $M^k=\begin{bmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{bmatrix}$.

From Lemma (2.2),

if $R(C) \subseteq R(A^k)$ and A is right k-regular then C=CXA _____(3.3) for each right k-g-inverse X of A.

We claim that $m=\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ is a right k-g-inverse of M.

$$M^k m M = \begin{bmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A^k XA & 0 \\ CA^{k-1} XA & 0 \end{bmatrix}$$

Since A is right k-regular by Definition (2.1), $A^k XA = A^k$. From (3.3), C=CXA.

Therefore, $M^k mM = \begin{bmatrix} A^k XA & 0 \\ CA^{k-1} XA & 0 \end{bmatrix}$

$$= \begin{bmatrix} A^k XA & 0 \\ CXAA^{k-1} XA & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A^k XA & 0 \\ CXA^k XA & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A^k & 0 \\ CXA^k & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A^k & 0 \\ CXAA^{k-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{bmatrix} = M^k$$

Hence M is right k-regular.

Conversely, $M = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right k-regular and by Lemma (2.1), $R(C) \subseteq R(A^k) \Rightarrow$ $C = XA^k$ for some $X \in \mathcal{F}_n$.

Hence $M = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$

$$= \begin{bmatrix} A & 0 \\ XA^k & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ XA^{k-1} & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

$$= UA'$$

Where $U = \begin{bmatrix} I & 0 \\ XA^{k-1} & 0 \end{bmatrix}$ and $A' = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$.

Since M is right k-regular $\Rightarrow UA'$ is right k-regular.

Hence $(UA')^k M^- UA' = (UA')^k$ where M^- is a right k-g-inverse of UA'.

Since $(UA')^k = U(A')^k$, $U(A')^k M^- UA' = U(A')^k$ _____ (3.4).

For $U^- = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $U^- U = I$ and U^- is a right k-g-inverse of U, premultiplying (3.4) by U^-,

$U^- U(A')^k M^- UA' = U^- U(A')^k \Rightarrow (A')^k (M^- U)A' = (A')^k$. Thus A' is right k-regular.

Hence A is right k-regular.

(ii) Can be proved in the same manner.

Lemma 3.2:

For A,B$\in \mathcal{F}_{mn}$, the following statements hold:

(i) $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is k-regular $\Leftrightarrow \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$ is k-regular.

(ii) $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is k-regular $\Leftrightarrow \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}$ is k-regular.

Proof:

From Theorem (2.1), M is k-regular $\Leftrightarrow$ PMP^T is k-regular for some permutation matrix P.

(i) $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is k-regular $\Leftrightarrow P\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}P^T$ is k-regular for some permutation matrix P

$$\Leftrightarrow \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix} \text{ is k-regular}$$

(ii) Can be proved in the same manner.

Theorem 3.1:

Let M be of the form (3.1).

(i) If either $R(C)\subseteq R(A^k)$ and $R(B)\subseteq R(D^k)$ (or) $C(B)\subseteq C(A^k)$ and $C(C)\subseteq C(D^k)$, then M can be expressed as sum of two k-regular block matrices.

(iii)If either $R(C)\subseteq R(A^k)$, $C(C)\subseteq C(D)$, $C(B)\subseteq C(A)$ and $R(B)\subseteq R(D^k)$, then M can be expressed as sum of two k-regular triangular block matrices.

Proof:

From (3.1), M=$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with the diagonal blocks A and D are k-regular.

(i) Since A is k-regular and $R(C)\subseteq R(A^k)$ by Lemma(3.1), $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right k-regular.

Since D is k-regular and $R(B)\subseteq R(D^k)$ by Lemma(3.1), $\begin{bmatrix} D & 0 \\ B & 0 \end{bmatrix}$ is right k-regular.

Hence by Lemma (3.2), $\begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}$ is right k-regular.

Thus M = $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}$ is sum of two right k-regular block matrices.

Similarly, if $C(B)\subseteq C(A^k)$ and $C(C)\subseteq C(D^k)$ then M = $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}$ is sum of two left k-regular block matrices. Thus (i) holds.

(ii) From Theorem (2.3), if A and D are k-regular with the conditions $R(C)\subseteq R(A^k)$ and $C(C)\subseteq C(D)$ then $\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ is k-regular.

From Theorem (2.4), if A and D are k-regular with the conditions $C(B)\subseteq C(A)$ and $R(B)\subseteq R(D^k)$ then $\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$ is k-regular.

Hence $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$ is sum of two k-regular triangular block matrices. Thus (ii) holds.

Theorem 3.2:

Let M be of the form (3.1) with $R(C)\subseteq R(A^k)$, $C(C)\subseteq C(D^k)$, $C(B)\subseteq C(A^k)$ and $R(B)\subseteq R(D^k)$, the schur complements M/A and M/D are fuzzy matrices, then M is k-regular and $m = \begin{bmatrix} X + XBYCX & XBY \\ YCX & Y \end{bmatrix}$ _____ (3.5)

is a k-g- inverse of M for some k-g inverses X of A and Y of D respectively.

Proof:

Since A is k-regular with $R(C)\subseteq R(A^k)$ and $C(B)\subseteq C(A^k)$ by Lemma(2.2), C=CXA, B=AXB for each k-g-inverse X of A.

Since M/A is a fuzzy matrix by (3.2), it follows that CXB is invariant for all choices of k-g-inverses X of A and D=D+CXB.

Now, under the given conditions, M can be expressed as M=ULV where

$U = \begin{bmatrix} I & 0 \\ CX & I \end{bmatrix}$, $L = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $V = \begin{bmatrix} I & XB \\ 0 & I \end{bmatrix}$.

Let us define $L^- = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ where X is a k-g-inverse of A and Y is a k-g-inverse of D.

On computation, we see that $VL^-U = V\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}U = m$ defined in (3.5).

By using induction on k, let us prove that M is k-regular.

For k =1, the result is precisely Theorem (3.1) of [2].

For k =2,

$M^2mM = (ULV)^2(VL^-U)(ULV)$ .Since U and V are idempotent matrices,

$M^2mM = (ULV)^2L^-(ULV) = M^2L^-M$

Let $M^2 = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ then $\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}$.

Hence $P = A^2+BC$ and $Q = AB+BD$ _____ (3.6).

$M^2mM = M^2L^-M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$$= \begin{bmatrix} PXA+QYC & PXB+QYD \\ RXA+SYC & RXB+SYD \end{bmatrix}$$

Now, we claim that $M^2mM = M^2L^-M = M^2$.
First we prove that the 11th block of M^2L^-M and that of M^2 are equal.
For this, it is enough to prove that $A^2+BC = PXA+QYC$.
By induction hypothesis, the given conditions reduce to the following:
A is 2-regular $\Rightarrow A^2XA=A^2$ by Definition (2.1)_____ (3.7).
$C(C)\subseteq C(D^2) \Rightarrow C=DYC$ by Lemma(2.2) _____(3.8).
$R(C)\subseteq R(A^2) \Rightarrow C=CXA$ by Lemma(2.2) _____(3.9).
By (3.2), M/D is a fuzzy matrix $\Rightarrow A=A+BYC$ _____(3.10).
By using equations (3.6) to (3.10) yields,

$$\begin{aligned} PXA+QYC &= A^2XA+BCXA+ABYC+BDYC \\ &= A^2+BC+ABYC+BC \\ &= A^2+ABYC+BC \\ &= A(A+BYC)+BC \\ &= A.A+BC \\ &= A^2+BC \end{aligned}$$

Thus, the 11th block of M^2L^-M and the 11th block of M^2 are equal.
Similarly, it can be verified that the remaining blocks of M^2L^-M and M^2 are equal.
Hence M is 2-regular.
Assume that $M^{k-1} L^- M = M^{k-1}$, then
$M^k L^- M = M(M^{k-1} L^- M) = MM^{k-1} = M^k$.
Hence $M^kmM = M^kL^-M = M^k$. Thus M is k-regular and m is a k-g-inverse of M.
Hence the theorem.

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Received: November, 2009