Further Results on the Generalised Growth Properties of Functions Analytic in a Unit Disc

Sanjib Kumar Datta

Department of Mathematics
University of North Bengal
Darjeeling, Pin-734013, West Bengal, India
sanjib.kr_datta@yahoo.co.in

Eleja Jerin

Department of Mathematics
Kalitala Diar R.J.K. High School
Kalital Diar, P.O.-Berhampore
Dist.-Murshidabad, PIN-742101, West Bengal, India

Abstract

Some comparative growth properties relating to relative generalised Nevanlinna order of an analytic function with respect to an entire function in a unit disc are studied in the paper.

Mathematics Subject Classification: 30D35, 30D30

Keywords: Growth, relative generalised Nevanlinna order, analytic function, entire function, unit disc

1 Introduction, Definitions and Notations.

Let \( f \) be an analytic function in the unit disc \( U = \{ z : |z| < 1 \} \). Juneja and Kapoor [2] introduced the notion of Nevanlinna order (Nevanlinna lower order) of an analytic function \( f \) in \( U \). Banerjee and Datta [1] extended this notion and defined the relative Nevanlinna order (relative Nevanlinna lower order) of an analytic function \( f \) with respect to an entire function \( g \). In the line of Banerjee and Datta [1] we may give the following definitions:
Definition 1 Let $T_f(r) = T(r, f)$ denote the Nevanlinna’s characteristic function of $f$. The relative generalised Nevanlinna order $\rho_g^{[l]}(f)$ and relative generalised Nevanlinna lower order $\lambda_g^{[l]}(f)$ of an analytic function $f$ in $U$ with respect to another entire function $g$ are defined in the following way:

$$\rho_g^{[l]}(f) = \limsup_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{-\log(1 - r)} \quad \text{and} \quad \lambda_g^{[l]}(f) = \liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{-\log(1 - r)},$$

where $\log^{[l]} x = \log \left(\log^{[l-1]} x\right)$ for $l = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

For $l = 2$, the quantities $\rho_g^{[2]}(f) = \tilde{\rho}_g(f)$ and $\lambda_g^{[2]}(f) = \tilde{\lambda}_g(f)$ are respectively called relative Nevanlinna hyper order and relative Nevanlinna hyper lower order of an analytic function $f$ in $U$ with respect to another entire function $g$.

The following definition is also well known.

Definition 2 Two entire functions $f$ and $g$ are said to be asymptotically equivalent if there exists $l$, $0 < l < \infty$ such that $\frac{F(r)}{G(r)} \to l$ as $r \to \infty$ and in this case we write $f \sim g$. If $f \sim g$ then clearly $g \sim f$.

In the paper we establish some results based on relative generalised Nevanlinna order of an analytic function $f$ in the unit disc $U = \{z : |z| < 1\}$. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [4].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

In the line of Lahiri and Banerjee [3] we may state the following lemmas without proof.

Lemma 1 If $f$, $g$ and $h$ be three entire functions such that $g \sim h$ then $\rho_g^{[l]}(f) = \rho_h^{[l]}(f)$ and $\lambda_g^{[l]}(f) = \lambda_h^{[l]}(f)$.

Lemma 2 If $f$, $g$ and $h$ be three entire functions such that $f \sim h$ then $\rho_g^{[l]}(f) = \rho_g^{[l]}(h)$ and $\lambda_g^{[l]}(f) = \lambda_g^{[l]}(h)$.

Lemma 3 If $f$, $g$, $h$ and $k$ be four entire functions such that $g \sim h$ and $f \sim k$ then $\rho_f^{[l]}(g) = \rho_k^{[l]}(h) = \rho_f^{[l]}(h) = \rho_k^{[l]}(g)$ and $\lambda_f^{[l]}(g) = \lambda_k^{[l]}(h) = \lambda_f^{[l]}(h) = \lambda_k^{[l]}(g)$.
3 Theorems.

In this section we present the main results of the paper.

Theorem 1 Let \( f \) be analytic function and \( g, h \) be two entire functions in \( U \) such that \( g \sim h \) and \( 0 < \rho_g^{[l]}(f) < \infty \). Then

\[
\liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leq 1 \leq \limsup_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_h^{-1} T_f(r)},
\]

where \( l = 1, 2, 3, \ldots \).

Proof. From the definition of relative generalised Nevanlinna order of an analytic function with respect to an entire function we get for a sequence of values of \( (\frac{1}{1-r}) \) tending to infinity

\[
\log^{[l]} T_h^{-1} T_f(r) \geq \left[ \rho_h^{[l]}(f) - \varepsilon \right] \left[ -\log(1-r) \right]
\]

and for all sufficiently large values of \( (\frac{1}{1-r}) \)

\[
\log^{[l]} T_h^{-1} T_f(r) \leq \left[ \rho_h^{[l]}(f) + \varepsilon \right] \left[ -\log(1-r) \right].
\]

Also for all large values of \( (\frac{1}{1-r}) \),

\[
\log^{[l]} T_g^{-1} T_f(r) \leq \left[ \rho_g^{[l]}(f) + \varepsilon \right] \left[ -\log(1-r) \right].
\]

Again for a sequence of values of \( (\frac{1}{1-r}) \) tending to infinity

\[
\log^{[l]} T_g^{-1} T_f(r) \geq \left[ \rho_g^{[l]}(f) - \varepsilon \right] \left[ -\log(1-r) \right].
\]

Now from (1) and (3) it follows for a sequence of values of \( (\frac{1}{1-r}) \) tending to infinity that

\[
\frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leq \frac{\rho_g^{[l]}(f) + \varepsilon}{\rho_h^{[l]}(f) - \varepsilon}.
\]

As \( \varepsilon (> 0) \) is arbitrary we get from above that

\[
\liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leq \frac{\rho_g^{[l]}(f)}{\rho_h^{[l]}(f)}.
\]

Now as \( g \sim h \), in view of Lemma 1 we obtain from (5) that

\[
\liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leq 1.
\]
Again combining (2) and (4) we get for a sequence of values of \( r \) tending to infinity,
\[
\frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \geq \frac{\rho^{[l]}_g (f) - \varepsilon}{\rho^{[l]}_h (f) + \varepsilon}.
\]
Since \( \varepsilon (> 0) \) is arbitrary it follows from above that
\[
\limsup_{r \to 1} \frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \geq \frac{\rho^{[l]}_g (f)}{\rho^{[l]}_h (f)}. \tag{7}
\]
Now as \( g \sim h \), in view of Lemma 1 we obtain from (7) that
\[
\limsup_{r \to 1} \frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \geq 1. \tag{8}
\]
Thus the theorem follows from (6) and (8).

**Theorem 2** Let \( f, h \) be two analytic functions and \( g \) be entire function in an unit disc \( U = \{ z : |z| < 1 \} \) such that \( f \sim h \) and \( 0 < \rho^{[l]}_g (f) < \infty \). Then
\[
\liminf_{r \to 1} \frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \leq 1 \leq \limsup_{r \to 1} \frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)},
\]
where \( l = 1, 2, 3, \ldots \).

**Proof.** From the definition of relative generalised Nevanlinna order we get for a sequence of values of \( \left( \frac{1}{1-r} \right) \) tending to infinity,
\[
\log^{[l]} T^{-1}_g T_h(r) \geq \left[ \rho^{[l]}_g (h) - \varepsilon \right] [- \log (1 - r)]. \tag{9}
\]
Now from (3) and (9) it follows for a sequence of values of \( \left( \frac{1}{1-r} \right) \) tending to infinity,
\[
\frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \leq \frac{\rho^{[l]}_g (f)}{\rho^{[l]}_h (h) - \varepsilon}.
\]
As \( \varepsilon (> 0) \) is arbitrary we obtain that
\[
\liminf_{r \to 1} \frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \leq \frac{\rho^{[l]}_g (f)}{\rho^{[l]}_h (h)}. \tag{10}
\]
Since \( f \sim h \), in view of Lemma 2 we obtain from (10) that
\[
\liminf_{r \to 1} \frac{\log^{[l]} T^{-1}_g T_f(r)}{\log^{[l]} T^{-1}_h T_f(r)} \leq 1. \tag{11}
\]
Again for all large values of \((\frac{1}{1-r})\),
\[
\log[l] T_g^{-1} T_h(r) \leq [\rho_g[l] (h) + \varepsilon] [-\log (1 - r)].
\] (12)
So combining (4) and (12) we get for a sequence of values of \((\frac{1}{1-r})\) tending to infinity,
\[
\frac{\log[l] T_g^{-1} T_f(r)}{\log[l] T_g^{-1} T_h(r)} \geq \frac{\rho_g[l] (f)}{\rho_g[l] (h)} - \varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary it follows that
\[
\limsup_{r \to 1} \frac{\log[l] T_g^{-1} T_f(r)}{\log[l] T_g^{-1} T_h(r)} \geq \frac{\rho_g[l] (f)}{\rho_g[l] (h)}.
\] (13)
Thus the theorem follows from (11) and (14).

**Theorem 3** Let \(f, h\) be two analytic functions and \(g, k\) be two entire functions in the unit disc \(U\) such that \(f \sim h\) and \(g \sim k\). Also let \(0 < \rho_g[l] (f) < \infty\). Then
\[
\liminf_{r \to 1} \frac{\log[l] T_g^{-1} T_f(r)}{\log[l] T_k^{-1} T_h(r)} \leq 1 \leq \limsup_{r \to 1} \frac{\log[l] T_g^{-1} T_f(r)}{\log[l] T_k^{-1} T_h(r)},
\]
where \(l = 1, 2, 3, \ldots \).

**Proof.** From the definition of relative generalised Nevanlinna order we get for a sequence of values of \((\frac{1}{1-r})\) tending to infinity
\[
\log[l] T_k^{-1} T_h(r) \geq \left[\rho_k[l] (h) - \varepsilon\right] [-\log (1 - r)]
\] (15)
and for all sufficiently large values of \((\frac{1}{1-r})\)
\[
\log[l] T_k^{-1} T_h(r) \leq \left[\rho_k[l] (h) + \varepsilon\right] [-\log (1 - r)].
\] (16)
Now from (3) and (15) it follows for a sequence of values of \((\frac{1}{1-r})\) tending to infinity,
\[
\frac{\log[l] T_g^{-1} T_f(r)}{\log[l] T_k^{-1} T_h(r)} \leq \frac{\rho_g[l] (f) + \varepsilon}{\rho_k[l] (h) - \varepsilon}.
\]
As \( \varepsilon (>0) \) is arbitrary we get from above that

\[
\liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_k^{-1} T_h(r)} \leq \frac{\rho_g^{[l]} (f)}{\rho_k^{[l]} (h)}.
\] (17)

Now as \( f \sim h \) and \( g \sim k \), in view of Lemma 3 we obtain from (17) that

\[
\liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_k^{-1} T_h(r)} \leq 1.
\] (18)

Again combining (4) and (16) we get for a sequence of values of \( \left( \frac{1}{1-r} \right) \) tending to infinity,

\[
\frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_k^{-1} T_h(r)} \geq \frac{\rho_g^{[l]} (f) - \varepsilon}{\rho_k^{[l]} (h) + \varepsilon}.
\]

Since \( \varepsilon (>0) \) is arbitrary it follows from above that

\[
\limsup_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_k^{-1} T_h(r)} \geq \frac{\rho_g^{[l]} (f)}{\rho_k^{[l]} (h)}.
\] (19)

Now as \( f \sim h \) and \( g \sim k \), in view of Lemma 3 we obtain from (19) that

\[
\limsup_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log^{[l]} T_k^{-1} T_h(r)} \geq 1.
\] (20)

Thus the theorem follows from (18) and (20)

In view of Lemma 3 one may also prove the following theorem in the line of Theorem 3.

**Theorem 4** Let \( f, h \) be two analytic functions and \( g, k \) be two entire functions in an unit disc \( U = \{ z : |z| < 1 \} \) such that \( f \sim h \) and \( g \sim k \). Also let \( 0 < \rho_g^{[l]} (f) < \infty \). Then

\[
\liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_h(r)}{\log^{[l]} T_k^{-1} T_f(r)} \leq 1 \leq \limsup_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_h(r)}{\log^{[l]} T_k^{-1} T_f(r)}.
\]

The proof is omitted.
References


Received: November, 2009