On the Properties of k-Fibonacci Numbers

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Abstract
In this paper, we obtain some new identities for k-Fibonacci numbers. Moreover the identities including generating functions for k-Fibonacci numbers have been obtained by Binet's Formula, also divisibility properties of these numbers have been investigated.

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1 Introduction

In recent years, Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art. For the beauty and rich applications of these numbers and their relatives one can see Koshy’s book and the nature. Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature(e.g. see [1-5] ). The well-known Fibonacci \{F_n\} sequence is defined as

\[ F_0 = 0, \ F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \quad for \quad n \geq 2 \]

where \( F_n \) denotes the \( n \)th Fibonacci number.

In [2], these general \( k \)-Fibonacci numbers \( \{F_{k,n}\}_{n=0}^{\infty} \) were found by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge(4TLE) partition. Many properties of these
numbers are obtained directly from elementary matrix algebra. In [3], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [4], authors defined $k$-Fibonacci hyperbolic functions similar to hyperbolic functions and Fibonacci hyperbolic functions. They deduced some properties of $k$-Fibonacci hyperbolic functions related with the analogous identities for the $k$-Fibonacci numbers. Finally, authors studied 3-dimensional $k$-Fibonacci spirals with a geometric point of view in [5]. Several properties of these new $k$-Fibonacci hyperbolic functions are studied in an easy way. In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [6]. $m$-extension of the Fibonacci and Lucas $p-$ numbers are defined in [7]. Afterwards, the continuous functions for the $m$-extension of the Fibonacci and Lucas $p$-numbers using the generalized Binet formulas. In [8], Stakhov and Rozin showed that the formulas are similar to the Binet formulas given for the classical Fibonacci numbers, also defined to be of generalized Fibonacci and Lucas numbers or Fibonacci and Lucas $p$-numbers.

In this paper, we obtain new identities for $k$-Fibonacci numbers. Moreover, the identities including generating functions for $k$-Fibonacci numbers have been obtained by Binet’s Formula, also divisibility properties of these numbers have been investigated.

2 The $k$-Fibonacci numbers and some properties

The $k$-Fibonacci numbers defined by Falco’n and Plaza for any real number $k$ as follows. In this section, we give some properties of the $k$-Fibonacci numbers.

**Definition 1 (Falco’n and Plaza)** For any positive real number $k$, the $k$-Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for} \ n \geq 1 \quad (1)$$

The equation given by (1) is a second order difference equation with constant coefficients. Therefore, it has the characteristic equation;

$$r^2 = kr + 1 \quad (2)$$

The roots of the characteristic equation Eq.2 are $r_{1,2} = \frac{k \pm \sqrt{k^2 + 4}}{2}$. Note that, since $0 < k$, then

$$r_2 < 0 < r_1, \ |r_2| < r_1$$
On the properties of k-Fibonacci numbers

\[ r_1 + r_2 = k, \quad r_1r_2 = -1, \quad r_1 - r_2 = \sqrt{k^2 + 4}. \] (3)

Throughout this paper, \( F_{k,n} \) denotes the \( n^{th} \) k-Fibonacci number and \( r_1, r_2 \) are the roots of the characteristic equation Eq.2.

**Proposition 1** For any integer \( n \geq 1 \), we have,

\[ r_1^{n+2} = kr_1^{n+1} + r_1^n \]

and

\[ r_2^{n+2} = kr_2^{n+1} + r_2^n \]

where \( r_1, r_2 \) are the roots of the characteristic equation Eq.2.

**Proof.** Since \( r_1 \) and \( r_2 \) are the roots of the characteristic equation Eq.2, then

\[ r_1^2 = kr_1 + 1 \] (4)
\[ r_2^2 = kr_2 + 1 \]

Now, multiplying both sides of these equations by \( r_1^n \) and \( r_2^n \) respectively, we obtain the desired result. ■

**Proposition 2** The \( n^{th} \) k-Fibonacci number is given by

\[ F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \] (5)

where \( r_1, r_2 \) are the roots of the characteristic equation Eq.2 and \( r_1 > r_2 \).

**Proof.** We use the principle of mathematical induction (PMI) on \( n \). It is clear that the result is true for \( n = 0 \) and \( n = 1 \) by hypothesis. Assume that it is true for any \( i \) such that \( 0 \leq i \leq l + 1 \), then

\[ F_{k,i} = \frac{1}{r_1 - r_2} (r_1^i - r_2^i) \]

It follows from definition of the \( k \)-Fibonacci numbers and from Eq. 5

\[ F_{k,l+2} = kF_{k,l+1} + F_{k,l} = \frac{r_1^{l+2} - r_2^{l+2}}{r_1 - r_2} \]

Thus, the formula is true for any positive integer \( n \). Immediate consequences of Binet’s Formula given by Eq. 5, can be seen in the following identities. ■
Proposition 3  For any integer $n \geq 0$, we write,
\[
\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} = F_{k,2n}.
\]

Proof. By using Eq. 5,
\[
\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} = \sum_{i=0}^{n} \binom{n}{i} k^i \frac{r_1^i - r_2^i}{r_1 - r_2}
\]
\[
= \frac{1}{r_1 - r_2} \left[ \sum_{i=0}^{n} \binom{n}{i} (kr_1)^i - \sum_{i=0}^{n} \binom{n}{i} (kr_2)^i \right]
\]
and by summing up the geometric partial sums $\sum_{i=0}^{n} r_j^i$ for $j = 1, 2$ we obtain
\[
\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} = \frac{1}{r_1 - r_2} [(1 + kr_1)^n - (1 + kr_2)^n],
\]
where $r_1, r_2$ are the roots of the characteristic equation of $F_{k,n}$. Now by using Eq. 4, we get
\[
\sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i} = \frac{1}{r_1 - r_2} (r_1^{2n} - r_2^{2n})
\]
\[
= F_{k,2n}.
\]

Proposition 4  For arbitrary integers $n, m \geq 1$, we have
\[
\sum_{i=1}^{n} F_{k,mi} = \frac{F_{k,mn+m} - (-1)^m F_{k,mn} - F_{k,m}}{r_1^m + r_2^m - (-1)^m - 1}.
\]

Proof. By using of Binet’s Formula given in Eq. 5 and taking into account that $r_1 - r_2 = k$, $r_1 r_2 = -1$ it is obtained
\[
\sum_{i=1}^{n} F_{k,mi} = \sum_{i=1}^{n} \frac{r_1^m - r_2^m}{r_1 - r_2}
\]
\[
= \frac{F_{k,mn+m} - (-1)^m F_{k,mn} - F_{k,m}}{r_1^m + r_2^m - (-1)^m - 1}
\]
as desired.
For $k = 1$, we get
\[
\sum_{i=1}^{n} F_{mi} = \frac{F_{mn+m} - (-1)^m F_{mn} - F_m}{r_1^m + r_2^m - (-1)^m - 1}.
\]

In a similar way, the following identities can be proved.

i) If $j > m$, then
\[
\sum_{i=0}^{n} F_{k,mi+j} = \frac{F_{kmn+m+j} - (-1)^m F_{kmn,j} - F_{km,j}}{r_1^m + r_2^m - (-1)^m - 1}.
\]

ii) $\sum_{i=0}^{n} F_{k,i+j} = -\frac{1}{k} [-F_{kn+j} - F_{kn+1} + F_{kj} + F_{kj-1}]

iii) $F_{k,-n} = (-1)^{n+1} F_{kn} \quad n \geq 1$

**Proposition 5** The following equalities are valid for all $a, b, c \in N$:

i) $F_{k,a+b-1} = F_{k,a}F_{k,b} + F_{k,a-1}F_{k,b-1}$

ii) $F_{k,a+b-2} = \frac{1}{k} (F_{k,a}F_{k,b} - F_{k,a-2}F_{k,b-2})$

iii) $F_{k,a+b+c-3} = \frac{1}{k} (F_{k,a}F_{k,b}F_{k,c} + kF_{k,a-1}F_{k,b-1}F_{k,c-1} - F_{k,a-2}F_{k,b-2}F_{k,c-2})$

**Proof.**

i) It is clear from Eq.5

ii) From the definition of the $k$-Fibonacci sequence,
\[
F_{k,a+b-2} = F_{k,a}F_{k,b-1} + F_{k,a-1}F_{k,b-2} \\
= F_{k,a} \left( \frac{F_{k,b} - F_{k,b-2}}{k} \right) + \left( \frac{F_{k,a} - F_{k,a-2}}{k} \right) F_{k,b-2} \\
= \frac{1}{k} (F_{k,a}F_{k,b} - F_{k,a-2}F_{k,b-2})
\]
is obtained.

iii) Using the definition of the $k$-Fibonacci sequence, we have
\[
F_{k,a+b+c-3} = F_{k,a-1}F_{k,b+c-3} + F_{k,a-1}F_{k,b+c-2} \\
= F_{k,a-1} (F_{k,b-2}F_{k,c-2} + F_{k,b-1}F_{k,c-1}) + F_{k,a} \left( \frac{F_{k,b}F_{k,c} + F_{k,b-2}F_{k,c-2}}{k} \right).
\]

Now, after some algebra
\[
F_{k,a+b+c-3} = \frac{1}{k} (F_{k,a}F_{k,b}F_{k,c} + kF_{k,a-1}F_{k,b-1}F_{k,c-1} - F_{k,a-2}F_{k,b-2}F_{k,c-2}).
\]

**Proposition 6** For any integer $n \geq 1$,
\[
r_1^n = r_1 F_{kn} + F_{kn-1}.
\]
Proof. By using the Eq. 4 and taking \( r_1 + r_2 = k \) and \( r_1 r_2 = -1 \), we can write,

\[
\begin{align*}
  r_1^2 &= r_1 (k - r_2) = r_1 k - r_1 r_2 = r_1 k + 1 \\
  r_1^3 &= r_1 (r_1 k + 1) = r_1^2 k + r_1 = r_1 (k^2 + 1) + k \\
  \vdots
\end{align*}
\]

Continuing in this manner, we get

\[
r_1^n = r_1 F_{k,n} + F_{k,n-1}
\]

and proof is completed. Similarly, \( r_2^n = r_2 F_{k,n} + F_{k,n-1} \). \( \blacksquare \)

2.1 Divisibility Properties of the \( k \)-Fibonacci Numbers

In this section, we investigate the divisibility properties of the \( k \)-Fibonacci numbers.

Proposition 7 Any two consecutive \( k \)-Fibonacci numbers are relatively prime, that is \( (F_{k,n}, F_{k,n-1}) = 1 \).

Proof. Using the Euclidean algorithm with \( F_{k,n} \) as the original dividend and \( F_{k,n-1} \) as the original divisor. This yields following system of linear equations:

\[
\begin{align*}
  F_{k,n} &= kF_{k,n-1} + F_{k,n-2} \\
  F_{k,n-1} &= kF_{k,n-2} + F_{k,n-3} \\
  \vdots \\
  F_{k,3} &= kF_{k,2} + F_{k,1} \\
  F_{k,2} &= kF_{k,1} + 0.
\end{align*}
\]

Thus it follows by the Euclidean algorithm that \( (F_{k,n}, F_{k,n-1}) = F_{k,1} = 1 \). \( \blacksquare \)

Proposition 8 For \( n, m \in \mathbb{Z}^+ \), then \( F_{k,m} | F_{k,mn} \).

Proof. (By induction). The given statement is clearly true when \( n = 1 \). Now assume that it is true for all integers through \( l \) where \( l \geq 1 \) : \( F_{k,m} | F_{k,mi} \) for every \( i \), where \( 1 \leq i \leq l \).

To show that \( F_{k,m} | F_{k,m(l+1)} \), we invoke Eq. 6:

\[
F_{k,r+s} = F_{k,r-1}F_{k,s} + F_{k,r}F_{k,s+1} \tag{6}
\]

\[
F_{k,m(l+1)} = F_{k,ml+m} = F_{k,ml-1}F_{k,m} + F_{k,ml}F_{k,m+1}
\]

Since \( F_{k,m} | F_{k,ml} \), by the induction hypothesis, it follows that \( F_{k,m} | F_{k,m(l+1)} \). Thus, by the strong version of the PMI, the result is true for all integer \( n \geq 1 \). \( \blacksquare \)
Proposition 9 For $q, n \in \mathbb{Z}^+$, $(F_{k,qn-1}, F_{k,n}) = 1$.

Proof. Let $d = (F_{k,qn-1}, F_{k,n})$. Then $d \mid F_{k,qn-1}$ and $d \mid F_{k,n}$. Since $F_{k,n} \mid F_{k,qn}$ by proposition 10, $d \mid F_{k,qn}$. Thus $d \mid F_{k,qn-1}$ and $d \mid F_{k,qn}$. But $(F_{k,qn-1}, F_{k,n}) = 1$ by divisibility algorithm. Therefore $d \mid 1$, so $d = 1$. Thus $(F_{k,qn-1}, F_{k,n}) = 1$. □

Proposition 10 Let $m = qn + r$ for $m, n, q, r \in \mathbb{Z}^+$. Then

$$(F_{k,m}, F_{k,n}) = (F_{k,qn}, F_{k,r}).$$

Proof.

$$(F_{k,m}, F_{k,n}) = (F_{k,qn+r}, F_{k,n}) = (F_{k,qn+1}F_{k,r} + F_{k,qn}F_{k,r+1}, F_{k,n}) \text{ by Eq.6} = (F_{k,qn-1}F_{k,r}, F_{k,n}) = (F_{k,r}, F_{k,n}) \text{ by proposition 11}$$

□

The next theorem shows that the greatest common divisor (gcd) of two $k$-Fibonacci numbers is always $k$-Fibonacci number. The proof is easy from the Euclidean algorithm proposition 10 and this proposition.

Proposition 11 Let $m$ and $n$ be positive integers. Then,

$$(F_{k,m}, F_{k,n}) = F_{k,(m,n)}.$$

Proof. Suppose $m \geq n$. Applying the Euclidean algorithm with the same dividend and the divisor $m$, we get the following sequence of equations:

$$m = q_0n + r_1 \quad 0 \leq r_1 < n$$
$$n = q_1r_1 + r_2 \quad 0 \leq r_2 < r_1$$
$$r_1 = q_2r_2 + r_3 \quad 0 \leq r_3 < r_2$$
$$\vdots$$
$$r_{n-2} = q_{n-1}r_{n-1} + r_n \quad 0 \leq r_n < r_{n-1}$$
$$r_{n-1} = q_nr_n + 0$$

By proposition 12, we can write $(F_{k,m}, F_{k,n}) = (F_{k,n}, F_{k,r_1}) = (F_{k,r_1}, F_{k,r_2}) = \cdots = (F_{k,r_{n-1}}, F_{k,r_n})$. But $r_n \mid r_{n-1}$, so $F_{k,r_n} \mid F_{k,r_{n-1}}$, by proposition 10. Therefore $(F_{k,r_{n-1}}, F_{k,r_n}) = F_{k,r_n}$. Thus $(F_{k,m}, F_{k,n}) = F_{k,r_n}$. But by Euclidean algorithm, $r_n = (m, n)$; Therefore $(F_{k,m}, F_{k,n}) = F_{k,(m,n)}$. □

Proposition 12 For $m, n \in \mathbb{Z}^+$, If $(m, n) = 1$, then $F_{k,m}F_{k,n} \mid F_{k,mn}$.

Proof. By Proposition 10, $F_{k,m} \mid F_{k,mn}$ and $F_{k,n} \mid F_{k,mn}$. Therefore, $[F_{k,m}, F_{k,n}] \mid F_{k,mn}$. But $(F_{k,m}, F_{k,n}) = F_{k,(m,n)} = F_{k,1} = 1$, so $[F_{k,m}, F_{k,n}] = F_{k,m}F_{k,n}$. Thus $F_{k,m}F_{k,n} \mid F_{k,mn}$. □
2.2 Generating Functions for $k$-Fibonacci Sequences

In this section, some properties of the generating functions for the $k$-Fibonacci sequences are given.

**Proposition 13** Let $m$ and $n$ be integers and $F_{k,n}$ is the $n$th $k$-Fibonacci number. The following equalities are valid:

\[ i) \sum_{n=0}^{\infty} F_{k,n}x^n = \frac{x}{1-kx-x^2} \]

\[ ii) \sum_{n=0}^{\infty} F_{k,n+1}x^n = \frac{1}{1-kx-x^2} \]

\[ iii) \sum_{n=0}^{\infty} F_{k,mn}x^n = \frac{F_{k,m}x}{1-(r_1^m+r_2^m)+(-1)^m x^2} \]

\[ iv) \sum_{n=0}^{\infty} F_{k,rm+1}x^n = \frac{F_{k,-(-1)^m k} F_{k,r-m}x}{1-(r_1^m+r_2^m)+(-1)^m x^2} \]

\[ v) \sum_{n=0}^{\infty} F_{k,2n}x^n = \frac{xk}{1-(k^2+2)x+x^2} \]

\[ vi) \sum_{n=0}^{\infty} F_{k,2n+1}x^n = \frac{(1-x)k}{1-(k^2+2)x+x^2} \]

**Proof.** By using of Binet’s Formula given in Eq.5, i), ii), iii), iv), v) and vi) can be shown.

Let $k=1$ in v) and vi), then we obtain following equalities which we known from Fibonacci sequence

\[ \sum_{n=0}^{\infty} F_{2n} = \frac{x}{1-3x+2x^2}, \quad \sum_{n=0}^{\infty} F_{2n+1} = \frac{1-x}{1-3x+2x^2}. \]

3 Conclusions

This paper presents the results of some properties of $k$-Fibonacci numbers by using the Binet’s Formula. The conclusions arising from the work are as follows:

1. Have been obtained some new identities for $k$-Fibonacci numbers which like classical Fibonacci numbers.

2. Divisibility properties of $k$-Fibonacci numbers have been investigated.

3. The identities including generating functions for the $k$-Fibonacci numbers have been obtained. Also, this identities has wonderful application to graph theory.
References


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