Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 20, 963 - 976

Characterization of Besov Type Spaces for the Dunkl Type Operator on the Real Line

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Abstract

In this paper, we have defined subspaces of L^p by differences using the Dunkl type translation operators, that we call Besov-Dunkl type spaces. Finally, characterization of this space by the Dunkl type convolution is provided.

Mathematics subject classification: 46E30, 46F12, 44A15

Keywords: Dunkl type operators, Dunkl type transform Dunkl type convolution, Dunkl type translation operators, Besov-Dunkl type spaces

1. Introduction

In 1989, Dunkl [5] introduced Dunkl operators on the real line and are denoted by Λ_α, where α is a real parameter $>-\frac{1}{2}$. Besov spaces are defined by many authors from time to time (see [4, 12, 16]). In [1], [2], [8], Besov-Dunkl spaces are defined. In this paper we define Besov-Dunkl type spaces.

Definition 1.1: Let $b>0$, $1\le p<+\infty$, $1\le q\le+\infty$. The Besov-Dunkl type space denoted by $BD^{p,q}_{b,\alpha,\beta}$ is the subspace of functions $f\in L^p(\sigma_{\alpha,\beta})$ satisfying.

$$\int_0^{+\infty}\left(\frac{\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha,\beta}}{x^b}\right)^q\frac{dx}{x}<+\infty, if\ q<+\infty$$

and

$$\sup_{x\in(0,+\infty)} \frac{\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha,\beta}}{x^b} < +\infty, \; if \;\; q=+\infty$$

where $\sigma_{\alpha,\beta}$ is a weighted Lebesgue measure on R.

Now set

$$A=\left\{\phi\in S_*(R): \int_0^{+\infty}\phi(x)\,d\sigma_{\alpha,\beta}(x)=0\right\}$$

where $S_*(R)$ is the space of even Schwartz functions on R.

Definition 1.2: Given $\phi\in A$ we shall denote by $C^{p,q}_{\phi,b,\alpha,\beta}$ the subspace of functions $f\in L^p(\sigma_{\alpha,\beta})$ satisfying

$$\int_0^{\infty}\left(\frac{\|f *_{\alpha,\beta}\phi_t\|_{p,\alpha,\beta}}{t^b}\right)^q \frac{dt}{t} < +\infty; \; if \;\; q<+\infty$$

and

$$\sup_{t\in(0,+\infty)} \frac{\|f *_{\alpha,\beta}\phi_t\|_{p,\alpha,\beta}}{t^b} < +\infty \; if \; q=+\infty,$$

where

$$\phi_t(x)=\frac{1}{t^{2(\alpha-\beta)+3}}\,\phi\!\left(\frac{x}{t}\right) \; for\, all \;\; t\in(0,+\infty) \; and \;\; x\in R.$$

Definition 1.3: $\breve{B}\,D^{p,q}_{p,\alpha,\beta}$ is the subspace of functions $f\in L^p\left(\sigma_{\alpha-\beta}\right)$ satisfying

$$\int_0^{+\infty}\left(\|\tau_x(f)-f\|_{p,\alpha,\beta}\right)^q \frac{dx}{x} < +\infty, if \; q<+\infty$$

and

$$\sup_{x\in(0,+\infty)} \frac{\|\tau_x(f)-f\|_{p,\alpha,\beta}}{x^b} < +\infty, if \; q=+\infty.$$

2. Preliminaries

In this section we state some definitions and results, which are useful in the sequel.

Definition 2.1: The first order differential-difference type operator on the real line defined by

$$\Lambda_{\alpha,\beta}(f)(x) = D\, f(x) + \frac{2(\alpha-\beta+1)}{x}\left[\frac{f(x)-f(-x)}{2}\right],\ f \in C(R),$$

$(\alpha-\beta) > -1$ is called the Dunkl type operator, where $D = \frac{d}{dx}$, $C(R)$ denotes the space of C^{∞} functions on R.

Definition 2.2: For $\lambda \in$ R, the Dunkl type kernel $E_{\alpha,\beta}(\lambda)\, on\, R$ was introduced by Dunkl in [5] and is given by

$$E_{\alpha,\beta}(\lambda x) = j_{\alpha+\beta+1/2^{(i\lambda x)}} + \frac{\lambda x}{2(\alpha-\beta)+3}\ j_{\alpha-\beta+3/2}(i\lambda x),\ x \in R,$$

where

$$j_{\alpha-\beta+1/2^{(z)}} = 2^{\alpha-\beta+\frac{1}{2}}\Gamma\left(\alpha-\beta+3/2\right)\frac{J_{\alpha-\beta+1/2^{(z)}}}{z^{\alpha-\beta+1/2}} = \Gamma\left(\alpha-\beta+3/2\right)\sum_{n=0}^{+\infty}\frac{(-1)^n\,(z/2)^{2n}}{n!\,\Gamma(n+\alpha-\beta+3/2)}$$

is the normalized Bessel type function of the first kind and of order $\left(\alpha-\beta+\frac{1}{2}\right)$ (see [17]).

Let $\sigma_{\alpha,\beta}$ be the weighted Lebesgue measure on R given by

$$d\,\sigma_{\alpha,\beta}(x) = \frac{|x|^{2(\alpha-\beta+1)}}{2^{\alpha-\beta+\frac{3}{2}}\,\Gamma(\alpha-\beta+3/2)}\ dx\,.$$

For every $1 \le p < +\infty$, we denote by $L^p(\sigma_{\alpha,\beta})$, the space $L^p(R, d\sigma_{\alpha,\beta})$ and we use $\|\,.\,\|_{p,\alpha,\beta}$ as a shorthand for $\|\,.\,\|_{L^p(\sigma_{\alpha,\beta})}$.

We define Dunkl type transform for $f \in L^1(\sigma_{\alpha,\beta})$ as

$$F_{\alpha,\beta}(f)(x) = \int_R E_{\alpha,\beta}(-i\,x\,y)\ f(y)\,d\,\sigma_{\alpha,\beta}(y),\ x \in R\,.$$

Consider

$$W_{\alpha,\beta}(x,y,z) = \frac{\left(\Gamma(\alpha-\beta+3/2)\right)^2}{2^{\alpha-\beta-\frac{1}{2}}\sqrt{\pi}\,\Gamma(\alpha-\beta+1)}\left(1-b_{x,y,z}+b_{z,x,y}+b_{z,y,x}\right)\Delta_{\alpha,\beta}(x,y,z)\text{, for all } x,y,z \in R \qquad (2.1)$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy}; if\ x,y,z \in R-\{0\},\ z \in R \\ 0 \qquad ; \quad otherwise \end{cases}$$

and

$$\Delta_{\alpha,\beta} = \begin{cases} \dfrac{\left\{\left[\left(|x|+|y|^2\right)-z^2\right]\left[z^2-\left(|x|-|y|\right)^2\right]^{\alpha-\beta}\right\}}{|xyz|^{2(\alpha-\beta)+1}}, & if\ |z| \in S_{x,y} \\ 0 & , otherwise \end{cases}$$

where

$$S_{x,y} = \left[\left||x| - |y|\right|, |x| + |y|\right].$$

The kernel $W_{\alpha,\beta}$ is even and we have

$$W_{\alpha,\beta}(x,y,z) = W_{\alpha,\beta}(y,x,z) = W_{\alpha,\beta}(-x,z,y) = W_{\alpha,\beta}(-z,y,-x),$$

and

$$\int_R \left|W_{\alpha,\beta}(x,y,z)\ d\sigma_{\alpha,\beta}(z) \le 4.$$

We consider the signed measure $\mu_{x,y}$ on R defined by

$$d\mu_{x,y}(z) = \begin{cases} W_{\alpha,\beta}(x,y,z)\ d\sigma_{\alpha,\beta}(z), & if\ x,y \in R-\{0\} \\ d\delta_x(z) & ,if\ y=0 \\ d\delta_y(z) & ,if\ x=0 \end{cases}.$$

For $x, y \in R$ and f a continuous function on R, the Dunkl type translation operator τ_x is given by

$$\tau_x(f)(y) = \int_R f(z)\ d\mu_{x,y}(z)\ . \tag{2.2}$$

For $x \in R, \tau_x$ is a continuous linear operator from $C(R)$ into itself and for all $f \in C(R)$,where

$$\tau_x(f)(y) = \tau_y(f)(x),\ \tau_0(f)(x) = f(x),\ for\ x, y \in R,$$

And $C(R)$ denotes the space of C^∞ functions on R (see[9]). The operator τ_x can be extended to $L^p(\sigma_{\alpha,\beta}),\ 1 \le p \le +\infty$ and for $f \in L^p(\sigma_{\alpha,\beta})$,we have

$$\|\tau_x(f)\|_{p,\alpha,\beta} \le 4\|f\|_{p,\alpha,\beta} \tag{2.3}$$

(see [14], [15]).

Using the change of variable $z = \psi(x,y,z) = \sqrt{x^2 + y^2 - 2xy\cos\theta}$, we have

$$\tau_x(f)(y) = C_{\alpha,\beta} \int_0^\pi \left[f(\psi) + f(-\psi) + \frac{x+y}{\psi}(f(\psi) - f(-\psi)\right] d\nu_{\alpha,\beta}(\theta) \tag{2.4}$$

where

$$d\nu_{\alpha,\beta}(\theta) = (1-\cos\theta)\sin^{2(\alpha-\beta)+1}\theta\, d\theta$$

and

$$C_{\alpha,\beta} = \frac{\Gamma(\alpha-\beta+3/2)}{2\sqrt{\pi}\,\Gamma(\alpha-\beta+1)}.$$

From the generalized Taylor formula with integral remainder we have for $f \in C(R)$ and $x, y \in R$,

$$\left(\tau_x(f)-f\right)(y)=\int_{-|x|}^{|x|}\left[\frac{\operatorname{sgn}(x)}{2|x|^{2(\alpha-\beta+1)}}+\frac{\operatorname{sgn}(z)}{2|z|^{2(\alpha-\beta+1)}}\right]\tau_z(\Lambda_{\alpha,\beta f})(y)\,d\sigma_{\alpha,\beta}(z)$$

(see [11, Theorem 2, p. 349]).

The Dunkl type convolution $f *_{\alpha,\beta} g$ of two continuous functions f *and* g on R with compact support is defined by

$$\left(f *_{\alpha,\beta} g\right)(x)=\int_R \tau_x(f)(-y)\,g(y)\,d\sigma_{\alpha,\beta}(y), \quad x\in R.$$

One can easily verify that $*_{\alpha,\beta}$ is commutative and associative.

We have the following results:

(i) Assume that $p,q,r\in[1,+\infty)$ satisfying $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ (the Young condition). Then the map $(f,g)\to f *_{\alpha,\beta} g$ defined on $C_c(R)\times C_c(R)$, extends to a continuous map from $L^p(\sigma_{\alpha,\beta})\times L^q(\sigma_{\alpha,\beta})$ to $L^r(\sigma_{\alpha,\beta})$ and we have

$$\left\| f *_{\alpha,\beta} g\right\|_{r,\alpha,\beta} \le 4\|f\|_{p,\alpha,\beta}\|g\|_{q,\alpha,\beta} \tag{2.5}$$

(ii) For all $f\in L^1(\sigma_{\alpha,\beta})$ and $g\in L^2(\sigma_{\alpha,\beta})$, we have

$$F_{\alpha,\beta}(f *_{\alpha,\beta} g)=F_{\alpha,\beta}(f)\,f_{\alpha,\beta}(g) \tag{2.6}$$

(iii) For $f\in L^1(\sigma_{\alpha,\beta}), g\in L^p(\sigma_{\alpha,\beta}), 1\le p<\infty$,

$$\tau_t(f *_{\alpha,\beta} g)=\tau_t(f) *_{\alpha,\beta} g = f *_{\alpha,\beta} \tau_t(g),\ t\in R. \tag{2.7}$$

3. Characterization of Besov-Dunkl type Spaces

Let $b>0, 1\le p<+\infty$ and $1\le q\le+\infty$. We say that a measurable function f *on* R is in the Besov-Dunkl type space $BD^{p,q}_{b,\alpha,\beta}$ if $f\in L^p(\sigma_{\alpha,\beta})$ with

$$\int_0^{+\infty}\left[\frac{\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}}{x^b}\right]^q\frac{dx}{x}<+\infty, if\ q<+\infty$$

and

$$\sup_{x\in(0,+\infty)}\frac{\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}}{x^b}<+\infty, if\ q=+\infty.$$

From [10, Lemma 1, (ii)], it is clear that for $f\in L^p(\sigma_{\alpha,\beta})$ $f:R\to R^+$ defined by

$$f(x)=\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}$$

is measurable.

Lemma 3.1: Let $0<b<1,\ 1\le p<+\infty, 1\le q\le +\infty$ and

$$f\in L^p(\sigma_{\alpha,\beta}).\ \textit{If}\ \Lambda_{\alpha,\beta}(f)\in L^p(\sigma_{\alpha,\beta})\ \textit{then}\ f\in B\,D^{p,q}_{b,\alpha,\beta}.$$

Proof: By using the generalized Taylor formula, MinKowski's inequality for integrals and (2.3), we have

$$\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha,\beta}\le\|\tau_x(f)-f\|_{p,\alpha,\beta}+\|\tau_{-x}(f)-f\|_{p,\alpha,\beta}$$

$$\le C\left\|\Lambda_{\alpha,\beta}(f)\right\|_{p,\alpha,\beta}\int_{-x}^{x}\left[\frac{1}{2|x|^{2(\alpha-\beta+1)}}+\frac{1}{2|z|^{2(\alpha-\beta+1)}}\right]d\sigma_{\alpha,\beta}(z).$$

Hence for $x>0$, we can obtain

$$\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha,\beta}\le C\,x\left\|\Lambda_{\alpha,\beta}(f)\right\|_{p,\alpha,\beta}.$$

Now for $L>0$, it follows that

$$\int_0^{+\infty}\left[\frac{\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha,\beta}}{x^b}\right]^q\frac{dx}{x}\le C\int_0^{L}\left[x\left\|\Lambda_{\alpha,\beta}(f)\right\|_{p,\alpha,\beta}\right]^q\frac{dx}{x}+C\int_L^{+\infty}\left[\frac{\|f\|_{p,\alpha,\beta}}{x^b}\right]^q\frac{dx}{x}<+\infty.$$

We can prove the result when $q=+\infty$ by making use of usual modification. The proof is complete.

Remark: Let $0<b<1,\ 1\le p<+\infty\ and\ 1\le q\le+\infty$. By Lemma 3.1 we can infer that

(i) $S(R), C^1_c(R)\subset B\,D^{p,q}_{b,\alpha,\beta}$.

(ii) The functions g, h_n defined on R, by $g(x)=e^{-|x|}$ and $h_n(x)=\dfrac{x^n}{\cosh x}, n\in\mathbb{N}$ $are\ in\ B\,D^{p,q}_{b,\alpha,\beta}$.

Before proving the theorem of this paper we require the following lemmas:

Lemma 3.2: Let $\phi\in A, 1\le p<+\infty$ and $r>0,$ then there exists a constant $K>0$ such that for all $f\in L^p(\sigma_{\alpha,\beta})$ and $t>0,$ we have

$$\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}\le K\int_0^{+\infty}\min\left\{\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3},\left(\frac{t}{x}\right)^r\right\}\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha,\beta}\frac{dx}{x}. \tag{3.1}$$

Proof: For $t>0,$ we have

$$\int_0^{+\infty}\phi_t(x)\,d\sigma_{\alpha,\beta}(x)=\int_0^{+\infty}\phi(x)\,d\sigma_{\alpha,\beta}(x)=0$$

and

$$\left(\phi_t *_{\alpha,\beta} f\right)(y)=\int_R\phi_t(x)\,\tau_y(f)(-x)\,d\sigma_{\alpha,\beta}(x)$$

$$=\int_R \phi_t(x)\,\tau_y(f)(x)\,d\sigma_{\alpha,\beta}(x)\,.$$

Then for $y\in R$, we can have

$$2(\phi_t *_{\alpha,\beta} f)(y)=\int_R \phi_t(x)\left[\tau_y(f)(x)+\tau_y(f)(-x)-2f(y)\right]d\sigma_{\alpha,\beta}(x)$$

$$=2\int_0^{+\infty}\phi_t(x)\left[\tau_x(f)(y)+\tau_{-x}(f)(y)-2f(y)\right]d\sigma_{\alpha,\beta}(x)\,.$$

Using MinKowski's inequality for integrals, we obtain

$$\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}\le\int_0^{+\infty}\left|\phi_t(x)\right|\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}d\sigma_{\alpha,\beta}(x)$$

$$\le K\int_0^{+\infty}\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3}\left|\phi\left(\frac{x}{t}\right)\right|\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}\frac{dx}{x} \qquad (3.2)$$

$$\le K\int_0^{+\infty}\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3}\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}\frac{dx}{x}. \qquad (3.3)$$

As $\phi\in S_*(R)$, then for $r>0$ there exists a constant K such that

$$\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3+r}\left|\phi\left(\frac{x}{t}\right)\right|\le K\,.$$

By using (3.2) we can infer that

$$\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}\le K\int_0^{+\infty}\left(\frac{t}{x}\right)^{r}\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}\frac{dx}{x}.$$

Finally by using (3.3) and (3.4), we can obtain (3.1).
This completes the proof.

Lemma 3.3: Let $\phi\in A$ and $1<p<+\infty$, there exists a constant $K>0$ such that for all $f\in L^p(\sigma_{\alpha,\beta})$ and $x>0$, we have

$$\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}\le K\int_0^{+\infty}\min\left\{1,\frac{x}{t}\right\}\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}\frac{dt}{t}\,. \qquad (3.5)$$

Proof: For $0<\varepsilon<\delta<+\infty$, set

$$f_{\varepsilon,\delta}(y)=\int_\varepsilon^\delta\left(\phi_t *_{\alpha,\beta}\phi_t *_{\alpha,\beta} f\right)(y)\frac{dt}{t},\ y\in R\,.$$

By interchanging the orders of integration and using (2.7), we obtain

$$\tau_x(f_{\varepsilon,\delta})(y)=\int_\varepsilon^\delta\tau_x(\phi_t *_{\alpha,\beta}\phi_t *_{\alpha,\beta} f)(y)\frac{dt}{t}$$

$$=\int_{\varepsilon}^{\delta}\left(\tau_x(\phi_t)*_{\alpha,\beta}\phi_t*_{\alpha,\beta}f\right)(y)\frac{dt}{t}, y\in R, x\in(0,+\infty).$$

Now for $x\in(0,+\infty), y\in R,$ we have

$$(\tau_x(f_{\varepsilon,\delta})+\tau_{-x}(f_{\varepsilon,\delta})-2f_{\varepsilon,\delta})(y)=\int_{\varepsilon}^{\delta}\left[(\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t)*_{\alpha,\beta}\phi_t*_{\alpha,\beta}f\right](y)\frac{dt}{t}.$$

Making use of MinKowski's inequality for integrals and (2.5), we obtain

$$\begin{aligned}&\left\|(\tau_x(f_{\varepsilon,\delta})+\tau_{-x}(f_{\varepsilon,\delta})-2f_{\varepsilon,\delta}\right\|_{p,\alpha,\beta}\\ &\le\int_{\varepsilon}^{\delta}\left\|(\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t)*_{\alpha,\beta}\phi_t*_{\alpha,\beta}f\right\|_{p,\alpha,\beta}\frac{dt}{t}\\ &\le K\int_{\varepsilon}^{\delta}\left\|\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t\right\|_{1,\alpha,\beta}\left\|\phi_t*_{\alpha,\beta}f\right\|_{p,\alpha,\beta}\frac{dt}{t}.\end{aligned}\tag{3.6}$$

Now for $x,t\in(0,+\infty)$, we have

$$\begin{aligned}&\left\|\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t\right\|_{1,\alpha,\beta}\\ &=\int_R\left|\left[\int_R\phi_t(z)(d\mu_{x,y}(z)+d\mu_{x,y}(z)\right]-2\phi_t(y)\right|d\sigma_{\alpha,\beta}(y)\\ &=\int_R\left|\left[\int_R\phi\left(\frac{z}{t}\right)\left(d\mu_{x,y}(z)+d\mu_{x,y}(z)\right)\right]-2\phi\left(\frac{y}{t}\right)\right|t^{-2(\alpha-\beta)-3}\,d\sigma_{\alpha,\beta}(y).\end{aligned}$$

Using (2.1) and $z=z't$, we have

$$W_{\alpha,\beta}(x,y,z't)t^{2(\alpha-\beta)+3}=W_{\alpha,\beta}\left(\frac{x}{t},\frac{y}{t},z'\right).$$

Now from (2.2), we have

$$d\mu_{x,y}(z)=d\mu_{\frac{x}{t},\frac{y}{t}}(z')\ \textit{and}\ d\mu_{-x,y}(z)=d\mu_{\frac{-x}{t},\frac{y}{t}}(z').$$

Thus

$$\begin{aligned}&\left\|\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t\right\|_{1,\alpha,\beta}\\ &=\int_R\left|\left[\int_R\phi(z')\left(d\mu_{\frac{x}{t},\frac{y}{t}}(z')+d\mu_{\frac{-x}{t},\frac{y}{t}}(z')\right)\right]-2\phi\left(\frac{y}{t}\right)\right|t^{-2(\alpha-\beta)-3}\,d\sigma_{\alpha,\beta}(y)\\ &=\int_R\left|\left[\tau_{\frac{x}{t}}(\phi)\left(\frac{y}{t}\right)+\tau_{\frac{-x}{t}}(\phi)\left(\frac{y}{t}\right)\right]t^{-2(\alpha-\beta)-3}-2\phi_t(y)\right|d\sigma_{\alpha,\beta}(y)\\ &=\left\|\left(\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right)_t\right\|_{1,\alpha,\beta}\end{aligned}\tag{3.7}$$

$$=\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha,\beta}.$$

Since $\phi\in S_*(R),$ then using (2.4) and [7, Theorem 2.1], we can assert that

$$\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha,\beta}\leq K\frac{x}{t}\|\phi'\|_{1,\alpha,\beta}\leq K\frac{x}{t}.$$

On the other hand, by using (2.3), we have

$$\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha,\beta}\leq 10\,\|\phi\|_{1,\alpha,\beta}\leq K.$$

Thus we get

$$\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha,\beta}\leq K\,min\left\{1,\frac{x}{t}\right\}. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we obtain

$$\left\|\tau_x(f_{\varepsilon,\delta})+\tau_{-x}(f_{\varepsilon,\delta})-2f_{\varepsilon,\delta}\right\|_{p,\alpha,\beta}\leq K\int_{\varepsilon}^{\delta}min\left\{1,\frac{x}{t}\right\}\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}\frac{dt}{t}. \tag{3.9}$$

Now using (2.6), we can deduce that

$$\begin{aligned}\int_R(\phi*_{\alpha,\beta}\phi)(x)|x|^{2(\alpha-\beta+1)}dx&=2^{\alpha-\beta+3/2}\,\Gamma(\alpha-\beta+3/2)\,f_{\alpha,\beta}(\phi*_{\alpha,\beta}\phi)(0)\\&=2^{\alpha-\beta+3/2}\,\Gamma(\alpha-\beta+3/2)\left(f_{\alpha,\beta}(\phi)(0)\right)^2\\&=2^{\alpha-\beta+3/2}\,\Gamma(\alpha-\beta+3/2)\left(\int_R\phi(z)\,d\sigma_{\alpha,\beta}(z)\right)^2\\&=0.\end{aligned}$$

As $\phi*_{\alpha,\beta}\phi\in S(R),$ we can obtain

$$\int_R\left|\log|x|\right|\left|\phi*_{\alpha,\beta}\phi(x)\right||x|^{2(\alpha-\beta+1)}dx<\infty.$$

From [10, Theorem 3], using Calderon's reproducing formula for Dunkl operators, we have

$$\lim_{\substack{\varepsilon\to\infty\\ \delta\to+\infty}} f_{\in,\delta}=Kf,\ in\,L^p\,(\sigma_{\alpha,\beta}).$$

Thus from (2.3) and (3.9), we can obtain (3.5). This completes the proof.

Lemma 3.4: Let $0\leq\varepsilon,r<+\infty, and\ r>b>0,$ then there exist constants $K_{1,}\,K_2>0$ such that

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^b min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^r\right\}\right]\frac{dy}{y}\leq K_1,\ z\in(0,+\infty) \tag{3.10}$$

and

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^b \min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^{r}\right\}\right]\frac{dz}{z}\leq K_{2,}\ y\in(0,+\infty)\ . \tag{3.11}$$

Proof: We can write

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^b \min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^{r}\right\}\right]\frac{dy}{y}$$

$$= z^{-(b+\varepsilon)}\int_0^{z} y^{b+\varepsilon-1}\,dy + z^{r-b}\int_z^{+\infty} y^{b-r-1}\,dy \leq K_1,\ z\in(0,+\infty)$$

and

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^b \min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^{r}\right\}\right]\frac{dz}{z} = y^{b-r}\int_0^{y} z^{-b+r-1}\,d\,z + y^{b+\varepsilon}\int_y^{+\infty} z^{-b-\varepsilon-1}\,dz$$

$$\leq K_{2,}\ y\in(0,+\infty)\ .$$

This completes the proof.
Now we prove our main result of this paper.

Theorem 3.5:
(i) Let $1\leq p<+\infty,\ 1\leq q\leq+\infty$ *and* $b>0$ then we have for all $\phi\in A$

$$B\,D^{p,q}_{b,\alpha,\beta}\subset C^{p,q}_{\phi,b,\alpha,\beta}\ . \tag{3.12}$$

(ii) Let $1\leq p<+\infty,\ 1\leq q\leq+\infty$ *and* $0<b<1,$ then we have for all $\phi\in A$

$$B\,D^{p,q}_{b,\alpha,\beta}=C^{p,q}_{\phi,b,\alpha,\beta}\ . \tag{3.13}$$

Proof: Put $w_p^{\alpha,\beta}(f)(x)=\left\|\tau_x(f)+\tau_{-x}(f)-2f\right\|_{p,\alpha,\beta}$ for $f\in L^p(\sigma_{\alpha,\beta})$ *and* $q'=\dfrac{q}{q-1}$ the conjugate of q when $1<q<+\infty$.

(i) Suppose that $1\leq p<+\infty, 1\leq q\leq+\infty,\ \phi\in A,\ r>b$ and $f\in B\,D^{p,q}_{b,\alpha,\beta}$.

Case I: $q=1$ By (3.1) and Fubini's theorem, we have

$$\int_0^{+\infty}\frac{\left\|f *_{\alpha,\beta}\phi_t\right\|_{p,\alpha,\beta}}{t^b}\frac{dt}{t}\leq K\int_0^{+\infty}\int_0^{+\infty}\min\left\{\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3},\left(\frac{t}{x}\right)^{r}\right\}w_p^{\alpha,\beta}(f)(x)t^{-b-1}\,dt\,\frac{dx}{x}$$

$$\leq K\int_0^{+\infty} w_p^{\alpha,\beta}(f)(x)\left[\int_0^{+\infty}\min\left\{\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3},\left(\frac{t}{x}\right)^{r}\right\}t^{-b-1}\,dt\right]\frac{dx}{x}$$

$$\le K\int_0^{+\infty} W_p^{\alpha,\beta}(f)(x)\left[x^{-r}\int_0^x x^{r-b-1}\,dt + x^{2(\alpha-\beta)+3}\int_x^{+\infty} t^{-b-2(\alpha-\beta+2)}\right]\frac{dx}{x}$$

$$\le K\int_0^{+\infty}\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}\frac{dx}{x} < +\infty\,.$$

Hence $f\in C^{p,1}_{\phi,b,\alpha,\beta}$.

Case II: $q=+\infty$. By (3.1) we have

$$\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}\le K\left[\int_0^t\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3} w_p^{\alpha,\beta}(f)(x)\frac{dx}{x}+\int_t^{+\infty}\left(\frac{t}{x}\right)^r w_p^{\alpha,\beta}(f)(x)\frac{dx}{x}\right]$$

$$\le K\sup_{x\in(0,+\infty)}\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}\left[t^{-2(\alpha-\beta)-3}\int_0^t x^{2(\alpha-\beta+1)+b}\,dx + t^r\int_t^{+\infty} x^{-b-r-1}\,dx\right]$$

$$\le K\,t^b\sup_{x\in(0,+\infty)}\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}.$$

From this it is clear that $f\in C^{p,+\infty}_{\phi,b,\alpha,\beta}$.

Case III: $1<q<+\infty$. By (3.1), we have for $t>0$

$$\frac{\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}}{t^b}\le K\int_0^{+\infty}\left(\frac{x}{t}\right)^b\min\left\{\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3},\left(\frac{t}{x}\right)^r\right\}\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}\frac{dx}{x}.$$

Set

$$M(x,t)=\left(\frac{x}{t}\right)^b\min\left\{\left(\frac{x}{t}\right)^{2(\alpha-\beta)+3},\left(\frac{t}{x}\right)^r\right\}.$$

Now by using Holder's inequality and (3.10), we can obtain

$$\frac{\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}}{x^b}\le K\int_0^{+\infty}\left[M(x,t)\right]^{\frac{1}{q'}}\left[(M(x,t))^{\frac{1}{q}}\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}\right]\frac{dx}{x}$$

$$\le K\left\{\int_0^{+\infty}M(x,t)\left[\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}\right]^q\frac{dx}{x}\right\}^{\frac{1}{q}}.$$

By Fubini's theorem and (3.11), we have

$$\int_0^{+\infty}\left(\frac{\left\|\phi_t *_{\alpha,\beta} f\right\|_{p,\alpha,\beta}}{t^b}\right)^q\frac{dt}{t}\le K\int_0^{+\infty}\left(\frac{w_p^{\alpha,\beta}(f)(x)}{x^b}\right)^q\left(\int_0^{+\infty}M(x,t)\frac{dt}{t}\right)\frac{dx}{x}$$

$$\leq K \int_0^{+\infty} \left(\frac{w_p^{\alpha,\beta}(f)(x)}{x^b} \right)^q \frac{dx}{x} < +\infty.$$

This proves (i).

(ii) Let $f \in C_{\phi,b,\alpha,\beta}^{p,q}, \phi \in A, 0 < b < 1.$ For $1 < p < +\infty$ and $1 \leq q \leq +\infty,$ we have to show only that $f \in BD_{b,\alpha,\beta}^{p,q}.$

Case I: $q = 1$. By (3.5) and Fubini's theorem, we have

$$\int_0^{+\infty} \frac{w_p^{\alpha,\beta}(f)(x)}{x^b} \frac{dx}{x} \leq K \int_0^{+\infty}\int_0^{+\infty} \min\left\{1, \frac{x}{t}\right\} \|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta} \, x^{-b-1} \frac{dt}{t} dx$$

$$\leq K \int_0^{+\infty} \|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta} \left[\int_0^{+\infty} \min\left\{1, \frac{x}{t}\right\} x^{-b-1} \, dx \right] \frac{dt}{t}$$

$$\leq K \int_0^{+\infty} \|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta} \left[\frac{1}{t} \int_0^t x^{-b} \, dx + \int_t^{+\infty} x^{-b-1} dx \right] \frac{dt}{t}$$

$$\leq K \int_0^{+\infty} \frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \frac{dt}{t} < +\infty.$$

This proves the result for $q = 1$.

Case II: $q = +\infty$. By (3.5), we have

$$w_p^{\alpha,\beta}(f)(x) \leq K \left[\int_0^x \|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta} \frac{dt}{t} + \int_x^{+\infty} \frac{x}{t} \|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta} \frac{dt}{t} \right]$$

$$\leq K \sup_{t \in (0,+\infty)} \frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \left[\int_0^x t^{b-1} \, dt + x \int_x^{+\infty} t^{b-2} \, dt \right]$$

$$\leq K \, x^b \sup_{t \in (0,+\infty)} \frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b}.$$

Thus we conclude that $f \in BD_{b,\alpha,\beta}^{p,+\infty}$.

Case III: $1 < q < +\infty$. By (3.5) again, we have for $x > 0$

$$\frac{w_p^{\alpha,\beta}(f)(x)}{x^b} \leq K \int_0^{+\infty} \left(\frac{t}{x} \right)^b \min\left\{1, \frac{x}{t}\right\} \frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \frac{dt}{t}.$$

Put

$$N(x,t) = \left(\frac{t}{x} \right)^b \min\left\{1, \frac{x}{t}\right\}.$$

Using Holder's inequality and (3.10), we can obtain

$$\frac{w_p^{\alpha,\beta}(f)(x)}{x^b} \le K \int_0^{+\infty} [N(x,t)]^{\frac{1}{q}} \left[\left(N(x,t)\right)^{\frac{1}{q}} \frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \right] \frac{dt}{t}$$

$$\le K \left[\int_0^{+\infty} N(x,t) \left(\frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \right)^q \frac{dt}{t} \right]^{\frac{1}{q}}.$$

Now by using Fubini's theorem and (3.11), we have

$$\int_0^{+\infty} \left(\frac{W_p^{\alpha,\beta}(f)(x)}{x^b} \right)^q \frac{dx}{x} \le K \int_0^{+\infty} \left(\frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \right)^q \left(\int_0^{+\infty} N(x,t) \frac{dx}{x} \right) \frac{dt}{t}$$

$$\le K \int_0^{+\infty} \left(\frac{\|\phi_t *_{\alpha,\beta} f\|_{p,\alpha,\beta}}{t^b} \right)^q \frac{dt}{t} < +\infty .$$

This completes the proof of case III and hence the proof of (ii).

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Received: August, 2009