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Double Integral Identities and Zeta Functions

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Abstract. We develop new identities, in terms of Zeta functions and double integral representations, for sums of double binomial coefficients.

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1. Introduction

There are a number of results representing the summation of binomial coefficients in either integral or closed form. In particular cases Batir [1] and [2] found closed form representations for sums of the form $S(k,\alpha) = \sum_{n\geq 1} \frac{1}{n^k \binom{\alpha n}{n}}$, for $\alpha = 2$ and 3. Earlier Zucker [11] found $S(2,2) = \frac{1}{3}\zeta(2) = -\int_0^1 \frac{\ln\left(x^2-x+1\right)}{x}dx$. Krattenthaler and Rao [4] used Beta function methods to represent π in terms of binomial sums, Sofo [6], [7], [8], [9], used similar methods. In this paper we establish new results, by the use of the Beta function, to represent $\sum_{n\geq 1} \frac{t^n}{n^3 \binom{an+j}{j}\binom{bn+k}{k}}$ in integral form and for specific parameter values give new identities in terms of Zeta functions. Double integral representations of binomial sums, different than the ones presented in this paper, have also been given by various authors including Rhin and Viola [5] and Sondow [10]

2. SUMS AND ZETA FUNCTIONS

Theorem 1. *For $|t| \le 1$, a and $b \ge 0$, and $j, k \ge 0$, then*

$$\sum_{n\ge 1} \frac{t^n}{n^3 (an+jj)(bn+kk)} = -ab \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^k}{xy} \ln\left(1 - t x^a y^b\right) dx dy \tag{2.1}$$

for $\left|t x^a y^b\right| < 1$.

Proof. Consider

$$\sum_{n\ge 1} \frac{t^n}{n^3 (an+jj)(bn+kk)} = \sum_{n\ge 1} \frac{t^n \Gamma(j+1)\Gamma(k+1)\Gamma(an+1)\Gamma(bn+1)}{n^3 \Gamma(an+j+1)\Gamma(bn+k+1)}$$
$$= ab \sum_{n\ge 1} \frac{t^n}{n} B(an, j+1) B(bn, k+1),$$

where the Beta function

$$B(s,z) = \int_0^1 w^{s-1} (1-w)^{z-1} dw = \frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)}$$

for $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(z) > 0$, and the Gamma function

$$\Gamma(z) = \int_0^\infty w^{z-1} e^{-w} dw, \quad \text{for } \operatorname{Re}(z) > 0.$$

Now

$$ab \int_0^1 \frac{(1-x)^j}{x} \sum_{n\ge 1} \frac{t^n x^{an}}{n} B(bn, k+1) dx$$
$$= ab \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^k}{xy} \sum_{n\ge 1} \frac{t^n \left(x^a y^b\right)^n}{n} dx dy$$
$$= -ab \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^k \ln\left(1 - t x^a y^b\right)}{xy} dx dy$$

for $\left|t x^a y^b\right| < 1$. ∎

Corollary 1. *For $a=b=1,\ t=\frac{1}{2},\ j=0$ and for integer $k\geq 1$, then*

$$\begin{aligned}
&\sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n}{n^3\,(n+kk)} \\
(2.2)\qquad &= -\int_0^1\int_0^1 \frac{(1-y)^k\ln\left(1-\frac{1}{2}xy\right)}{xy}dxdy \\
&= \frac{7}{8}\zeta(3)-\frac{1}{2}\left(H_k^{(1)}+\ln 2\right)\zeta(2) \\
&\quad +\frac{1}{6}\left((\ln 2)^3+3(\ln 2)^2\cdot H_k^{(1)}+3\ln 2\left\{\left(H_k^{(1)}\right)^2+H_k^{(2)}\right\}\right) \\
(2.3)\qquad &\quad -\sum_{r=1}^{k}\frac{2^r\,(kr)}{r^2}B(-1;r+1,-r),
\end{aligned}$$

where the incomplete Beta function

$$\begin{aligned}
B(z;\alpha,\beta) &= \int_0^z u^{\alpha-1}(1-u)^{\beta-1}\,du, \quad \textit{for } \operatorname{Re}(\alpha)>1 \textit{ and } \operatorname{Re}(\beta)>1, \\
&= \frac{z^\alpha}{\alpha}\ {}_2F_1(\alpha,1-\beta,1+\alpha;z)
\end{aligned}$$

in terms of the Gauss hypergeometric function.

Proof. Put

$$\begin{aligned}
(2.4)\qquad \sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n}{n^3\,(n+kk)} &= \sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n k!}{n^3\,(n+1)_{k+1}} = \sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n k!}{n^3\prod_{r=1}^k(n+r)} \\
&= \sum_{n\geq 1}\frac{k!\left(\frac{1}{2}\right)^n}{n^3}\sum_{r=1}^k\frac{A_r}{n+r},
\end{aligned}$$

where

$$(2.5)\qquad A_r=\lim_{n\to(-r)}\left\{\frac{n+r}{(n+1)_{k+1}}\right\}=\frac{(-1)^{r+1}}{(k-r)!\,(r-1)!}=\frac{(-1)^{r+1}}{k!}r\,(kr).$$

From (2.4)

$$(2.6)\qquad \sum_{n\geq 1}\frac{k!\left(\frac{1}{2}\right)^n}{n^3}\sum_{r=1}^k\frac{(-1)^{r+1}\,r}{k!\,(n+r)}(kr)=\sum_{r=1}^k(-1)^{r+1}\,r\,(kr)\sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n}{n^3\,(n+r)}.$$

By further partial fraction decomposition

$$(2.7)\qquad \sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n}{n^3(n+r)} = \sum_{n\geq 1}\left(\frac{1}{2}\right)^n\left[\frac{1}{rn^3}-\frac{1}{r^2n^2}+\frac{1}{r^2n(n+r)}\right]$$
$$= -\frac{1}{24r^3(r+1)}\Big[24\cdot 2^r(-1)^{r+1}B(-1,r+1,-r) + (r+1)\left\{2r\pi^2(1+r\ln 2) - 12r(\ln 2)^2 - 24\ln 2 - 4r^2(\ln 2)^3 - 21r^2\zeta(3)\right\}\Big].$$

From (2.6) and using (2.7), we have

$$\sum_{r=1}^{k}\frac{(-1)^r\binom{k}{r}}{24r^2(r+1)}\Big[24\cdot 2^r(-1)^{r+1}B(-1,r+1,-r) + (r+1)\left\{2r\pi^2(1+r\ln 2)-12r(\ln 2)^2-24\ln 2-4r^2(\ln 2)^3-21r^2\zeta(3)\right\}\Big]$$
$$= \frac{7}{8}\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\cdot\zeta(3) + \frac{1}{2}\sum_{r=1}^{k}\frac{(-1)^r\binom{k}{r}}{r}\{1+r\ln 2\}\zeta(2)$$
$$+\frac{1}{6}\sum_{r=1}^{k}\frac{(-1)^r\binom{k}{r}}{r^2}\left\{r^2(\ln 2)^3+3r(\ln 2)^2+6\ln 2\right\}$$
$$-\sum_{r=1}^{k}\frac{2^r\binom{k}{r}}{r^2}B(-1;r+1,-r)$$

and by an extensive amount of algebraic simplification, with the aid of Mathematica, we obtain (2.3). ∎

The hypergeometric representation of

$$\sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n}{n^3\binom{n+k}{k}} = \frac{1}{2(k+1)}\ {}_4F_3\left[\begin{matrix}1,1,1,1\\2,2,2+k\end{matrix}\,\middle|\,\frac{1}{2}\right].$$

The degenerate case $k=0$ results in

$$\sum_{n\geq 1}\frac{\left(\frac{1}{2}\right)^n}{n^3} = \frac{7}{8}\zeta(3)-\frac{1}{2}\zeta(2)+\frac{1}{6}(\ln 2)^3.$$

Remark 1. *Similar calculations allow us to evaluate many other specific cases such as*

$$\sum_{n\geq 1}\frac{1}{n^3(n+kk)} = \zeta(3) - H_k^{(1)}\zeta(2) + \sum_{r=1}^{k}\frac{(-1)^r(kr)}{r^2}H_r^{(1)} \quad \text{and}$$

$$\sum_{n\geq 1}\frac{1}{n^3(2n+kk)} = \zeta(3) - 2H_k^{(1)}\zeta(2) + \sum_{r=1}^{k}\frac{(-1)^r(kr)}{r^2}H_{\frac{r}{2}}^{(1)}.$$

Corollary 2. *For $t=1$, $b=1$, $j=1$, $a=2$ and integer $k\geq 1$, then*

(2.8)
$$\sum_{n\geq 1}\frac{1}{n^3(2n+1)(n+kk)} = -2\int_0^1\int_0^1\frac{(1-x)(1-y)^k\ln(1-x^2y)}{xy}dxdy$$

$$= \zeta(3) - \left(2+H_k^{(1)}\right)\zeta(2) + 8k(1-\ln 2)B\left(\frac{1}{2},k\right)$$

(2.9)
$$+\sum_{r=1}^{k}\frac{(-1)^r(kr)}{r^2(2r-1)}H_r^{(1)},$$

where $B(\cdot,\cdot)$ is the classical Beta function.

Proof. Consider the expansion

$$\sum_{n\geq 1}\frac{1}{n^3(2n+1)(n+kk)} = \sum_{n\geq 1}\frac{k!}{n^3(2n+1)(n+1)_{k+1}}$$

$$= \sum_{n\geq 1}\frac{k!}{n^3(2n+1)}\sum_{r=1}^{k}\frac{A_r}{n+r}$$

$$= \sum_{n\geq 1}(-1)^{r+1}r(kr)\sum_{r=1}^{k}\frac{1}{n^3(2n+1)(n+r)}.$$

Now proceeding as in Corollary 1 we have the result (2.9). ∎

With the aid of a mathematical computer package, we can write

$$\sum_{n\geq 1} \frac{1}{n^3 (2n+1)(n+kk)}$$

$$= \frac{4}{3(1+k)} {}_3F_2 \left[\begin{matrix} 1,1,\frac{3}{2} \\ \frac{5}{2}, 2+k \end{matrix} \middle| 1 \right]$$

$$+ \frac{1}{1+k} {}_4F_3 \left[\begin{matrix} 1,1,1,1 \\ 2,2,2+k \end{matrix} \middle| 1 \right] + 2H_k^{(1)} - 2\zeta(2) \tag{2.10}$$

$$= -2 \int_0^1 \int_0^1 \frac{(1-x)(1-y)^k \ln(1-x^2 y)}{xy} dxdy$$

$$= \frac{1}{2} \int_0^1 \frac{(1-y)^k}{y^{3/2}} \left[4\arctan\left(y^{1/2}\right) + 2y^{1/2}(-2+\ln(1-y)) + y^{1/2}\psi''(y) \right] dy, \tag{2.11}$$

where $\psi''(y)$ is the Polygamma function. It can be argued that neither (2.10) nor (2.11) are as computationally efficient as (2.9).

The degenerate case of (2.8) for $k=0$ gives us the result

$$\sum_{n\geq 1} \frac{1}{n^3 (2n+1)} = \zeta(3) - 2\zeta(2) - 8\ln 2 + 8.$$

Corollary 3 now follows.

Corollary 3. *For $a=b=1$, $t=1$ and integer $j=k\geq 1$, then*

$$\sum_{n\geq 1} \frac{1}{n^3 (n+kk)^2} = -\int_0^1 \int_0^1 \frac{[(1-x)(1-y)]^k \ln(1-xy)}{xy} dxdy \tag{2.12}$$

$$= \zeta(3) + \sum_{r=1}^{k} (kr)^2 \left[2\left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} - \frac{3}{r} \right] \zeta(2)$$

$$+ \sum_{r=1}^{k} \frac{(kr)^2}{r} \left[H_r^{(2)} + \frac{3H_r^{(1)}}{r} - 2H_r^{(1)} \left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} \right]. \tag{2.13}$$

Proof. Expand

$$\sum_{n\geq 1} \frac{1}{n^3 (n+kk)^2} = \sum_{n\geq 1} \frac{(k!)^2}{n^3 \left((n+1)_{k+1}\right)^2} = \sum_{n\geq 1} \frac{(k!)^2}{n^3 \prod_{r=1}^{k} (n+r)^2}$$

$$= \sum_{n\geq 1} \frac{(k!)^2}{n^3} \sum_{r=1}^{k} \left[\frac{A_r}{n+r} + \frac{B_r}{(n+r)^2} \right]$$

where

$$B_r = \lim_{n\to(-r)} \left[\frac{(n+r)^2}{\prod_{r=1}^{k}(n+r)^2} \right] = \left(\frac{r}{k!}(kr) \right)^2$$

and

$$A_r = \lim_{n\to(-r)} \frac{d}{dn} \left[\frac{(n+r)^2}{\prod_{r=1}^{k}(n+r)^2} \right]$$
$$= -2\left(\frac{r}{k!}(kr) \right)^2 \left[H_{k-r}^{(1)} - H_{r-1}^{(1)} \right].$$

Now, by interchanging sums we have

(2.14)

$$\sum_{n\geq 1} \frac{1}{n^3 (n+kk)^2} = \sum_{r=1}^{k} (r(kr))^2 \left[\sum_{n\geq 1} \left\{ \frac{1}{n^3(n+r)^2} - \frac{2\left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\}}{n^3(n+r)} \right\} \right]$$

we can use

$$\sum_{n\geq 1} \frac{1}{n^3(n+r)} = \frac{\zeta(3)}{r} - \frac{\zeta(2)}{r^2} + \frac{H_r^{(1)}}{r^3}$$

and also the fact that

$$\sum_{n\geq 1} \frac{1}{n^3(n+r)^2} = \frac{3}{r^3 n(n+r)} - \frac{3}{r^3(n+r)^2} - \frac{2}{r^3 n^2} + \frac{1}{r^2 n^3}$$
$$= \frac{\zeta(3)}{r^2} - \frac{3\zeta(2)}{r^3} + \frac{H_r^{(2)}}{r^3} + \frac{3H_r^{(1)}}{r^4},$$

so that (2.14) becomes

$$\sum_{n\geq 1} \frac{1}{n^3(n+kk)^2} = \sum_{r=1}^{k} (r(kr))^2 \left[\frac{\zeta(3)}{r^2} - \frac{3\zeta(2)}{r^3} + \frac{H_r^{(2)}}{r^3} + \frac{3H_r^{(1)}}{r^4} \right]$$
$$- 2\sum_{r=1}^{k} (r(kr))^2 \left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} \left[\frac{\zeta(3)}{r} - \frac{\zeta(2)}{r^2} + \frac{H_r^{(1)}}{r^3} \right]$$
$$= \sum_{r=1}^{k} (r(kr))^2 \left[\frac{1}{r^2} - \frac{2}{r}\left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} \right] \zeta(3)$$
$$+ \sum_{r=1}^{k} (r(kr))^2 \left[-\frac{3}{r^3} + \frac{2}{r^2}\left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} \right] \zeta(2)$$
$$+ \sum_{r=1}^{k} (r(kr))^2 \left[\frac{H_r^{(2)}}{r^3} + \frac{3H_r^{(1)}}{r^4} - \frac{2}{r^3}\left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} H_r^{(1)} \right]$$

and upon simplification, we obtain (2.13) and using the fact that, see [3],

$$\sum_{r=1}^{k} (kr)^2 \left[\alpha - 2\alpha r \left\{H_{k-r}^{(1)} - H_{r-1}^{(1)}\right\}\right] = \alpha, \qquad \alpha \neq 0.$$

∎

Remark 2. *In terms of hypergeometric functions, we can write*

$$\sum_{n\geq 1} \frac{1}{n^3 (n+44)^2} = \frac{1}{25}\ {}_4F_3 \left[\begin{array}{c} 1,1,1,1 \\ 2,6,6 \end{array} \middle| 1 \right] = \zeta(3) - \frac{175}{4}\zeta(2) + \frac{61175}{864}.$$

again which is not as computationally efficient as (2.13).

References

[1] N. Batir, On the series $\sum_{k=1}^{\infty} \binom{3k}{k}^{-1} k^{-n} x^k$, *Proc. Indian Acad. Sci.* (Math. Sci.), **115**(4), (2005), 371-381.

[2] N. Batir, Integral representations of some series involving $\binom{2k}{k}^{-1} k^{-n}$ and some related series. *Appl. Math. Comput.*, **147, (**2004**),** 645-667.

[3] K. Driver, H. Prodinger, C. Schneider and J.A.C. Weideman, Padé approximations to the logarithm II: Identities, recurrences and symbolic computation. *Ramanujan J.* **11**, (2006), 139-158.

[4] C. Krattenthaler, and K. S. Rao, Automatic generation of hypergeometric identities by the beta integral method. *J. Comput. Math. Appl.,* **160,** (2003), 159-173.

[5] G. Rhin and C. Viola, On the irrationality measure of $\zeta(2)$. *Annales del'institut Fourier.* **43,** (1993), 85-109.

[6] A. Sofo, Harmonic numbers and double binomial coefficients, To appear, *Integral Transforms and Spec. Funct.*, (2009).

[7] A. Sofo. Integral forms of sums associated with Harmonic numbers, *Appl. Math. Comput.* **207,** (2009), 365-372.

[8] A. Sofo. Sums of derivatives of binomial coefficients, *Advances Appl. Math.* **42,** (2009), 123-134.

[9] A. Sofo. Computational Techniques for the Summation of Series, Kluwer Academic/ Plenum Publishers, New York, 2003.

[10] J. Sondow, Double integrals for Euler's constant and $\ln \frac{4}{\pi}$ and an analog of Hadjicosta's formula. *Amer. Math. Monthly,* **112,** (2005), 61-65.

[11] I. J. Zucker, On the series $\sum_{k=1}^{\infty} \binom{2k}{k}^{-1} k^{-n}$ and related sums. *J. Number Theory,* **20,** (1985), 92-102.

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