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Min-Flat Dimensions and Min-Flat Rings

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Abstract

In this paper, we introduce the concept of min-flat dimension and min-flat ring, and investigate some properties of min-flat modules and min-flat rings. Some equivalent characterizations of left PS rings are also given.

Keywords: min-flat module; min-flat ring; PS ring; FS ring; dimension

1 Introduction

The research of the dimension of the modules is very useful to investigate the properties of the modules. We are familiar with the flat dimensions, more information about flat dimensions can be found in [3], we exploit the apparent analogy between flat and min-flat modules by imitating the definition of flat dimensions of modules. For a ring R, it was proven that every R-module has a min-flat cover(see[6]). Accordingly, in this article we introduce the concept of min-flat module and min-flat ring, investigate some properties of min-flat modules and min-flat rings, and give some characterizations of PS rings. We also research the relation between min-flat modules and FS rings. Finally, we get some conditions such that the min-flat left R-modules are flat.

2 Preliminary Notes

Throughout this article, R is an associative ring with identity and all modules are unitary R-modules. $M_R({}_RM)$ denotes a right (left) R-module. For any module M, the character module M^+ is defined by $M^+ = Hom_Z(M, Q/Z)$. Let M and N be R-module, $M \bigotimes N$(resp. $Tor_n(M, N)$) means $M \bigotimes_R N$ (resp. $Tor_n^R(M, N)$) for an integer $n \geq 1$.

Definition 2.1 [6] *A right R-module M is called min-flat in case $Tor_1(M, R/I) = 0$ for any simple left ideal I of R.*

Definition 2.2 [6] *R is called a left min-coherent ring in case every simple left ideal of R is finitely presented.*

Lemma 2.3 [6] *The following are equivalent for a right R-module M:*
(1) M is min-flat.
(2) M^+ is min-injective.

Lemma 2.4 [6] *Let R be a left min-coherent ring, a left R-module M is mininjective if and only if its character module M^+ is a min-flat right R-module.*

Definition 2.5 [5] *A right R-module exact sequence $0 \to A \to B \to C \to 0$ is said to be min-pure exact if the sequence $0 \to A \bigotimes R/Ra \to B \bigotimes R/Ra \to C \bigotimes R/Ra \to 0$ is exact, for any $a \in R$ such that Ra is simple. A right R-module M is called min-pure-injective if for every min-pure exact sequence $0 \to A \to B \to C \to 0$, the sequence $Hom(B, M) \to Hom(A, M) \to 0$ is exact.*

Lemma 2.6 [5] *The following are equivalent for a right R-module M:*
(1) M is min-flat.
(2) Every right R-module exact sequence $0 \to A \to B \to M \to 0$ is min-pure exact.

Definition 2.7 [8] *A left R-module M is said to be mininjective if $Ext^1(R/I, M) = 0$ for any simple left ideal I of R.*

Definition 2.8 [4] *We define the torsion-free dimension tf(M) of a right R-module M as the smallest integer $n \geq 0$ such that $Tor_{n+k}(M, R/Rr) = 0$ for all $r \in R$ and $k \geq 1$.*

3 Main Results

Definition 3.1 *We define the min-$flat$ dimension mfd(M) of a right R-module M as the smallest integer $n \geq 0$ such that $Tor_{n+k}(M, R/Ra) = 0$ for any $a \in R$ such that Ra is simple, $k \geq 1$.*

The *global right min-$flat$ dimension* of R is the supremum of the *min-$flat$ dimension* of right R-modules.

From the definition it is evident that for any left R-module M, mfd(M)$\leq$ tfd(M)$\leq$ *weak dimension*(M)$\leq$ *projective dimension*(M) .

Lemma 3.2 *If the weak dimension of the simple left ideal Ra of R are at most $n-1$, then the global right min-flat dimension of R is $\leq n$.*

Proof *For any right R-module M, the exact sequence $0 \to Ra \to R \to R/Ra \to 0$ induce the exact sequence $0 = Tor_{n+1}(M,R) \to Tor_{n+1}(M,R/Ra) \to Tor_n(M,Ra) \to Tor_n(M,R) = 0$. We get the natural isomorphism*

$Tor_{n+1}(M,R/Ra) \cong Tor_n(M,Ra)$. By hypothesis, $Tor_n(M,Ra) = 0$, thus $Tor_{n+1}(M,R/Ra) = 0$, then the global right min-flat dimension of R is $\leq n$.

The next result characterizes the rings with right (and left) min-flat dimension is 0.

Theorem 3.3 *The following conditions on a ring R are equivalent.*

(1) Every right R-module is min-flat.
(2) Every R/aR is min-flat, where Ra is any simple left ideal.
(3) R/aR is projective, where Ra is any simple left ideal.
(4) $Tor_1(R/aR, R/Ra) = 0$, where Ra is any simple left ideal.
(5) $a = aba$ for some $b \in R$.

Proof (1) $\Leftrightarrow$ (2) *is by [6, Theorem 5.10].*
(2) $\Rightarrow$ (3) *is by [6, Corollary 3.3].*
(3) $\Rightarrow$ (4) *is obvious.*
(4) $\Rightarrow$ (5) *Let J be any right ideal of R, I be any left ideal of R, the exact sequence $0 \to J \to R \to R/J \to 0$ induce the following row exact diagram:*

$$\begin{tikzcd}
0 \arrow[r] & Tor(R/J,R/I) \arrow[r] \arrow[d, "\alpha"'] & J \bigotimes R/I \arrow[r] \arrow[d, "\beta"'] & R \bigotimes R/I \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & J \bigcap I/JI \arrow[r] & J/JI \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}$$

Since β, γ are isomorphisms, so α is a isomorphisms, that is to say

$Tor_1(R/J, R/I) \cong (J \bigcap I)/JI$ for any right ideal J and left ideal I. Thus hypothesis (4) is equivalent to the condition that $aR \bigcap Ra = aRa$, where Ra is any simple left ideal, hence $a = aba$ for some $b \in R$.
(5) $\Rightarrow$ (2) *is by [6, Corollary 3.3].*

In fact, the ring with global right (and left) min-flat dimension is 0, is a left (and right) universally mininjective ring (i.e. every left R-module is mininjective).

Our next goal is to investigate the rings whose global right min-flat dimension is 1.

Definition 3.4 *A ring R is called right min-flat if every (finitely generated) right ideal J_R of R is min-flat.*

Lemma 3.5 *A ring R is a right min-flat ring if and only if every simple left ideal of R is flat (i.e. R is a left FS ring[9]).*

Proof *There are natural isomorphisms $Tor_1(J, R/Ra) \cong Tor_2(R/J, R/Ra)$, $Tor_1(R/J, Ra) \cong Tor_2(R/J, R/Ra)$, where J is any right ideal of R, Ra is any simple left ideal of R. Hence for any $a \in R$ such that Ra is simple and for any right J of R, $Tor_1(J, R/Ra) = 0$ holds exactly if $Tor_1(R/J, Ra) = 0$. This says that R is a right min-flat ring if and only if every simple left ideal of R is flat.*

Corollary 3.6 *All the (finitely generated) right ideals of a ring are min-flat if and only if all simple left ideal are flat.*

The obvious question is: When are submodule of min-flat R-modules again are min-flat? The answer is immediate.

Corollary 3.7 *(1) A ring R is a right min-flat ring if and only if submodule of min-flat right R-module are again min-flat.*
(2) The global right min-flat dimension of a ring is ≤ 1 if and only if it is a min-flat ring.

Proof *(1) Let M_R be a min-flat module, and N a submodule of M_R. Note that Ra is flat by Lemma 3.5, thus the exact sequence $0 \to Ra \to R \to R/Ra \to 0$ implies the exactness of $0 = Tor_2(M/N, R) \to Tor_2(M/N, R/Ra) \to Tor_1(M/N, Ra) = 0$ so $Tor_2(M/N, R/Ra) = 0$, but the exact sequence $0 \to N \to M \to M/N \to 0$ implies the exactness of $0 = Tor_2(M/N, R/Ra) \to Tor_1(N, R/Ra) \to Tor_1(M, R/Ra) = 0$ so $Tor_1(N, R/Ra) = 0$. Therefore, N is min-flat. The converse is trivial.*
(2) Necessity is obvious, while sufficiency is a consequence of what has already been proved in the first part.

From this corollary and Lemma 3.5, we have the following result.

Corollary 3.8 *A ring R is a left FS ring if and only if every submodule of any min-flat right R-module is min-flat.*

Recall that a ring R is called a left PS ring if every simple left ideal of R is projective[7], equivalently if left socle of R is projective. It is obvious that every left PS ring is a left min-flat ring.

Theorem 3.9 *The following are equivalent for a right min-flat ring.*

(1) R is a right PS ring.
(2) Direct product of min-flat left R-module are again min-flat.
(3) R^I is a min-flat left R-module for any set I.

Proof (1) $\Rightarrow$ (2) *Let $(F_i)_{i\in I}$ be a family of min-flat left R-modules, we should prove that $\prod F_i$ is min-flat. Suppose aR is a simple right ideal of R, thus we have a commutative diagram:*

$$\begin{array}{ccc} aR\otimes\prod F_i & \longrightarrow & R\otimes\prod F_i \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ \prod(aR\bigotimes F_i) & \longrightarrow & \prod(R\bigotimes F_i) \end{array}$$

Since $(F_i)_{i\in I}$ are min-flat, so the bottom row is exact. By [2, Theorem 3.2.22] α,β are isomorphisms, thus the above row is exact, that is to say, $\prod F_i$ is min-flat.
(2) $\Rightarrow$ (3) *is trivial.*
(3) $\Rightarrow$ (1) *Let aR be a simple right ideal, since R^I is a min-flat left R-module, by [6, Theorem 4.5] aR is finitely presented, but R is a right min-flat ring, we have aR is flat. Thus aR is projective, therefore R is a right PS ring.*

Recall that a right R module is said to be absolutely pure if it is a pure submodule in every right R module containing it[1].

Theorem 3.10 *For any min-coherent ring R, the following conditions are equivalent.*

(1) Min-flat left R-modules are flat.
(2) Mininjective right R-modules are absolutely pure.
(3) Mininjective pure-injective right R-modules are injective.

Proof (1) $\Rightarrow$ (3) *Suppose M_R is mininjective pure-injective right R-module, then M^+ is min-flat by Lemma 2.4, thus M^+ is flat by hypothesis, therefore M is injective.*
(3) $\Rightarrow$ (1) *Suppose ${}_RM$ is min-flat, then M^+ is mininjective pure-injective by Lemma 2.3, thus M^+ is injective by hypothesis, therefore M is flat.*
(2) $\Rightarrow$ (3) *Let M be a mininjective pure-injective right R-module, and $E(M)$ its injective hull, by hypothesis, M is pure in $E(M)$, thus M is injective.*
(3) $\Rightarrow$ (2) *We prove that if every mininjective pure-injective right R-module is injective, then for a mininjective right R-module A, every exact sequence $0\to A\to B\to C\to 0$ of right R-module is pure exact. Using the pure embedding $\mu: A\to PE(A)$ of A in its pure-injective hull $PE(A)$, consider the push-out diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\mu}\big\downarrow & & {\scriptstyle\nu}\big\downarrow & & \|\big\downarrow & & \\ 0 & \longrightarrow & PE(A) & \xrightarrow{\alpha'} & X & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

It is easy to see that the pure-injective hull of a mininjective module is again mininjective, so by hypothesis $PE(A)$ is injective. We infer that the bottom row splits, thus $\alpha'\mu = \nu\alpha$ embeds A as a pure submodule in X, so αA is pure in B.

It is obvious that min-pure exact sequences is a generalization of pure exact sequences, so any min-pure-injective modules are pure-injective.

Theorem 3.11 *The following two properties of a ring R are equivalent.*

(1) In right R-modules, all min-pure submodules are pure.
(2) Pure-injective right R-modules are min-pure-injecctive.

Proof $(1) \Rightarrow (2)$ *is obvious.*
$(2) \Rightarrow (1)$ *Suppose $0 \to A \to B \to C \to 0$ is a min-pure exact of right R-module, ${}_RM$ is a left R-module. Since M^+ is pure-injectie, thus it is min-pure-injective, we have the exact sequence $0 \to Hom(C, M^+) \to Hom(B, M^+) \to Hom(A, M^+) \to 0$, This means $0 \to (C \bigotimes M)^+ \to (B \bigotimes M)^+ \to (A \bigotimes M)^+ \to 0$ is exact, equivalently the sequence $0 \to A \bigotimes M \to B \bigotimes M \to C \bigotimes M \to 0$ is exact. Since M can be any left R-module, thus the sequence $0 \to A \to B \to C \to 0$ is pure exact.*

Corollary 3.12 *If R is a left min-flat ring, then the injective dimension of any min-pure-injective left R-module is ≤ 1.*

Proof *For a left ideal J of R, let $0 \to M \to B \to J \to 0$ be an exact sequence. Since J is min-flat, by Lemma 2.6, M is a min-pure submodule of B. Since M is also min-pure-injective, then the sequence splits, hence $Ext^1(J, M) = 0$. Thus $Ext^2(R/J, M) = 0$ for all left ideal J, whence $id(M) \leq 1$.*

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