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An Analytical Method for Finding Critical Path in a Fuzzy Project Network

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Abstract

Critical path method (CPM) techniques have become widely recognized as valuable tools for the planning and scheduling of large projects. The aim of this paper is to present an analytical method for measuring the criticality in a fuzzy project network , where the duration time of each activity is represented by a trapezoidal fuzzy number. In this paper, we use a new defuzzification formula for trapezoidal fuzzy number and apply to the float time (slack time) for each activity in the fuzzy project network to find the critical path. The defuzzification formula used for critical path can not be applied to the trapezoidal fuzzy number having equal elements because that trapezoidal fuzzy number is a crisp number. The proposed method can overcome the drawback of the existing fuzzy CPM method. We use examples to compare our proposed method with the existing method. The comparison reveal that the method proposed in this paper is more effective in determining the activity criticalities and finding the critical path.

Mathematics Subject Classification: 90B10, 90B15, 90B50, 90C06

Keywords: Critical path method, Trapezoidal fuzzy number , fuzzy project network. Defuzzification

1. Introduction

The critical path is the one from the start of the project to finish of project where the slack times are all zeros. The purpose of the Critical path method(CPM) is to identify critical activities on the critical path so that resources may be concentrated on these activities in order to reduce project length time.

Besides, CPM has proved very valuable in evaluating project performance and identifying bottlenecks. Thus, CPM is a vital tool for the planning and control of complex projects.

According to the critical path, the decision-maker can control the time and the cost of the project and improve the efficiency of resource allocation to ensure the project quality. CPM has been used in business management, factory production, etc. [1,2]. The activity duration time often is an uncertain value so that the result of classical CPM computation can not properly match the real-world situation.

In [3], Dubois et al. extended the fuzzy arithmetic operations model to compute the latest starting time of each activity in a project network. In [4], Hapke et al. used fuzzy arithmetic operations to compute the earliest starting time of each activity in a project network. In [5], Yao et al. used signed distance ranking of fuzzy numbers to find critical path in a fuzzy project network. In [6], Chen et al. used defuzzification method to find possible critical paths in a fuzzy project network. Chanas and Zielinski [7] assume that the operation time of each activity can be represented as a crisp value, interval or a fuzzy number. Dubois et al. [8] assigns a different level of importance to each activity on a critical path for a randomly chosen set of activities. C.T.Chen et al. [9] proposed a method to deal with completion time management and the critical degrees of all activities for a project network.

In this paper, we used a new defuzzification formula for trapezoidal fuzzy number and applied to the float time (slack time) for each activity in the fuzzy project network to find the critical path. The defuzzification formula used for critical path can not be applied to the trapezoidal fuzzy number having equal numbers because that trapezoidal fuzzy number is a crisp number. The proposed method can overcome the drawback of the existing fuzzy CPM method [6]. In [6], trapezoidal fuzzy number having equal numbers is used as a trapezoidal number and obtained defuzzified value for it to find critical path but that type of fuzzy number is a crisp number. We used examples to compare our proposed method with the existing method [6]. The comparison reveal that the method proposed in this paper is more effective in determining the activity criticalities and finding the critical path.

2. Fuzzy concept

A fuzzy set can be mathematically constructed by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set [10,11]. This grade corresponds to the individual's similarity to the concept represented by the fuzzy set. The fuzzy number $\tilde{A}$ is a fuzzy set whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions [12] :

(i) $\mu_{\tilde{A}}(x)$ is piecewise continuous ;

(ii) $\mu_{\tilde{A}}(x)$ is a convex fuzzy subset;

(iii) $\mu_{\tilde{A}}(x)$ is the normality of a fuzzy subset, implying that for at least one element x_0 the membership grade must be 1, i.e. $\mu_{\tilde{A}}(x_0)=1$.

Definition 1: A fuzzy number with membership function in the form

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b, \\ \dfrac{c-x}{c-b}, & b \le x \le c, \\ 0, & otherwise \end{cases} \tag{1}$$

is called a triangular fuzzy number $\tilde{A} = (a,b,c)$.

Theorem 1 : Let $\tilde{A} = (a,b,c)$ be a triangular fuzzy number. Then

$$\text{Centroid}\,(\tilde{A}) = \frac{a+b+c}{3} \tag{2}$$

Proof : From the definition of the centroid method by using Fig.1,we may write the following :

$$\text{Centroid}\,(\tilde{A}) = \frac{\int_a^b \frac{x-a}{b-a}.x\,dx + \int_b^c \frac{c-x}{c-b}.x\,dx}{\int_a^b \frac{x-a}{b-a}\,dx + \int_b^c \frac{c-x}{c-b}.\,dx}$$

$$= \frac{(c-b)\int_a^b \left(x^2 - ax\right)dx + (b-a)\int_b^c \left(cx - x^2\right)dx}{(c-b)\int_a^b \left(x-a\right)dx + (b-a)\int_b^c \left(c-x\right)dx}$$

$$= \frac{a+b+c}{3}.$$

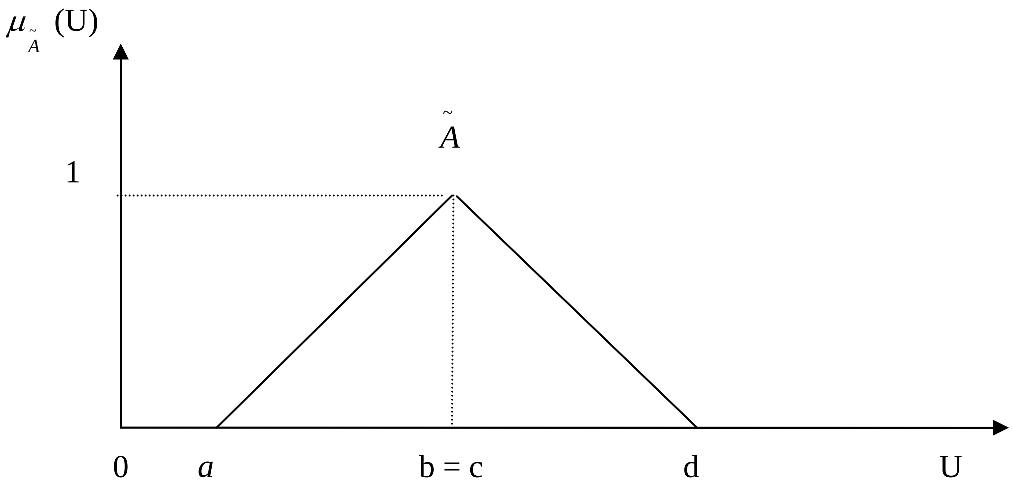

Fig.1. Membership function curve of triangular fuzzy number $\tilde{A}$.

Definition 2: A fuzzy number with membership function in the form

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b, \\ 1, & b \le x < c, \\ \dfrac{d-x}{d-c}, & c \le x \le d, \\ 0, & otherwise \end{cases} \tag{3}$$

is called a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$.

Theorem 2: Let $\tilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then

$$\text{Centroid}\,(\tilde{A}) = \frac{(c^2 + d^2 + cd) - (a^2 + b^2 + ab)}{3[(c+d) - (b+a)]} \tag{4}$$

Proof : From the definition of the centroid method we may write the following :

$$\text{Centroid}(\tilde{A}) = \frac{\int_a^b \frac{x-a}{b-a}.x\,dx + \int_b^c 1.x\,dx + \int_c^d \frac{d-x}{d-c}.x\,dx}{\int_a^b \frac{x-a}{b-a}\,dx + \int_b^c 1\,dx + \int_c^d \frac{d-x}{d-c}\,dx} \tag{5}$$

That Eq.(2) is valid becomes apparent once the corresponding integrals have been calculated.

A trapezoidal fuzzy number (1) may be represented by means of combinations of two triangular fuzzy numbers ($\tilde{A}_1$ and $\tilde{A}_3$) and a single rectangular fuzzy number ($\tilde{A}_2$) (Fig 2). By Theorem 1, we may write :

$$\text{Centroid}\ (\tilde{A}_1) = \frac{a+2b}{3} \tag{6}$$

$$\text{Centroid}\ (\tilde{A}_3) = \frac{2c+d}{3} \tag{7}$$

It is also seen that

$$\text{Centroid}\ (\tilde{A}_2) = \frac{b+c}{2} \tag{8}$$

The value of Centroid ($\tilde{A}$) may be calculated by means of Eq.(6) , Eq.(7) and Eq.(8) .

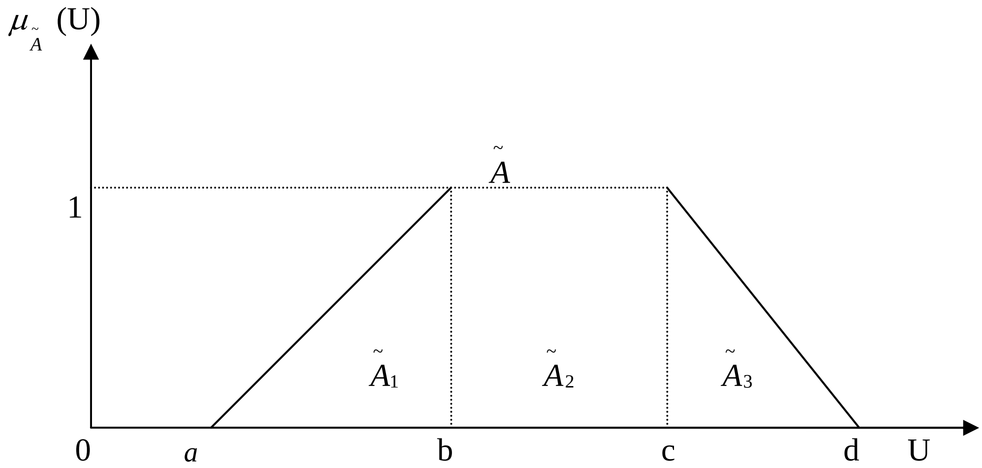

Fig.2. Membership function curve of trapezoidal fuzzy number $\tilde{A}$.

Let $\tilde{B}_1$ and $\tilde{B}_2$ be two trapezoidal fuzzy numbers parameterized by the quadruple (a_1 ,b_1,c_1,d_1) and (a_2 ,b_2,c_2,d_2), respectively. The simplified fuzzy number arithmetic operations between the trapezoidal fuzzy numbers $\tilde{B}_1$ and $\tilde{B}_2$ are as follows :

Fuzzy numbers addition $\oplus$:

$$(a_1,b_1,c_1,d_1) \oplus (a_2,b_2,c_2,d_2) = (a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2). \quad (9)$$

Fuzzy numbers subtraction Θ :

$$(a_1,b_1,c_1,d_1) \,\Theta\, (a_2,b_2,c_2,d_2) = (a_1-d_2, b_1-c_2, c_1-b_2, d_1-a_2) . \quad (10)$$

For example : Let $\tilde{B}_1$ and $\tilde{B}_2$ be two trapezoidal fuzzy numbers , where

$$\tilde{B}_1 = (16,20,22,24) \text{ and } \tilde{B}_2 = (3,4,5,6) \text{ . Then,}$$

$$\tilde{B}_1 \oplus \tilde{B}_2 = (16, 20, 22,24) \oplus (3,4,5,6) = (19, 24, 27,30)$$

$$\tilde{B}_1 \,\Theta\, \tilde{B}_2 = (16, 20, 22,24) \,\Theta\, (3,4,5,6) = (10,15,18,21)$$

3. Calculating fuzzy time values and critical path in a fuzzy project network

A fuzzy project network is an acyclic digraph, where the vertices represent events, and the direct edges represent the activities, to be performed in a project. Formally, A fuzzy project network is represented by N = (V,A,T). Let V = {v_1, v_2, …, v_n} be a set of vertices, where v_1 and v_n are the start and final events of the project, and each v_i belongs to some path from v_1 to v_n. Let $A \subset V \times V$ be the set of a directed edge $a_{ij} = (v_i,v_j)$, that represents the activities to be performed in the project. Activity a_{ij} is then represented by one, and only one, arrow with a tail event v_i , and a head event v_j . For each activity a_{ij}, a fuzzy number $\tilde{t}_{ij} \in T$ is defined, where $\tilde{t}_{ij}$ is the fuzzy time required for the completion of a_{ij} . A critical path is a longest path from v_1 to v_n, and an activity a_{ij} on a critical path is called a critical activity. Let $\tilde{E}_i$ and $\tilde{L}_i$ be the earliest fuzzy event time, and the latest fuzzy event time for event i, respectively. Let $\tilde{E}_j$ and $\tilde{L}_j$ be the earliest event time, and the latest event time for event j, respectively. Let D_j = {i / i∈V and $a_{ij} \in$ A} be a set of events obtained from event j ∈V and i < j. We then obtain $\tilde{E}_j$ using the following equations

$$\tilde{E}_j = \max_{i \in D_j} [\tilde{E}_i \oplus \tilde{t}_{ij}] \text{ and } \tilde{E}_1 = \tilde{L}_1 = 0. \quad (11)$$

Similarly, let $H_i = \{ j \,/\, j \in V$ and $a_{ij} \in A\}$ be a set of events obtained from event $i \in V$ and $i < j$. We obtain $\tilde{L}_i$ using the following equations

$$\tilde{L}_i = \min_{j \in H_i} [\, \tilde{L}_j \; \Theta \; \tilde{t}_{ij} \,] \text{and} \, \tilde{L}_n = \tilde{E}_n \qquad (12)$$

The interval $[\, \tilde{E}_i \;,\; \tilde{L}_j \,]$ is the time during which a_{ij} must be completed. When the earliest fuzzy event time and latest fuzzy event time have been obtained, we can calculate the total float of each activity. For activity i-j in a fuzzy project network, the total float $\tilde{T}_{ij}$ of the activity i-j can be computed as follows :

$$\tilde{T}_{ij} = \tilde{L}_j \; \Theta \; \tilde{E}_i \; \Theta \; \tilde{t}_{ij} \qquad (13)$$

Hence we can obtain the earliest fuzzy event time, latest fuzzy event time, and the total float of every activity by using (11)-(13).
We defuzzify the total float of each activity by Eq.(2) or Eq.(4) and find the critical path such that the sum of the total floats of the activities in the path is zero.

4. Examples

Example 1 :

Fig.3 shows the network representation of a fuzzy project network-I. Table I compares the original defuzzified value and proposed defuzzified value for the total float of each activity in the fuzzy project network.

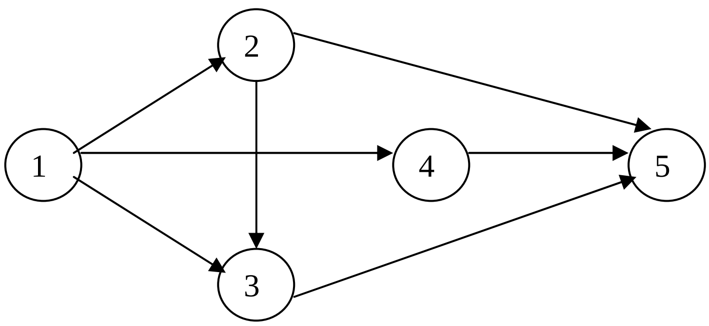

Fig.3. Fuzzy project network-I

Table I : Defuzzified values of slack time for fuzzy project **-I**

Activity	Fuzzy activity time	Slack time (a,b,c,d)	Original Defuzzified value $\frac{a+b+c+d}{4}$	Proposed Defuzzified value using Eq.(4)
1-2	(10,15,15,20)	(-160,-60,60,160)	0	0
1-3	(30,40,40,50)	(-130,-35,75,170)	20	20
2-3	(30,40,50,60)	(-160,-60,60,160)	0	0
1-4	(15,20,25,30)	(-110,-20,95,185)	37.5	37.5
2-5	(60,100,150,180)	(-100,-10,100,190)	45	45
3-5	(60,100,150,180)	(-160,-60,60,160)	0	0
4-5	(60,100,150,180)	(-110,-20,95,185)	37.5	37.5

The paths in fuzzy project network-I are 1-2-3-5 ,1-2-5 ,1-3-5, and 1-4-5 . The Critical path for fuzzy project network-I is 1-2-3-5.

Example2 :

Fig.4 shows the network representation of a fuzzy project network-II. Table II compares the original defuzzified value and proposed defuzzified value for the total float of each activity in the fuzzy project network.

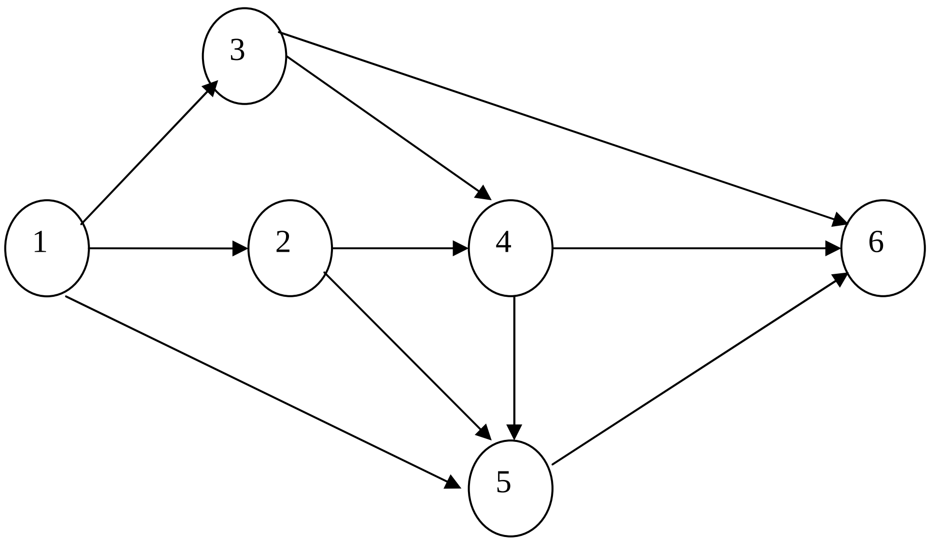

Fig.4. Fuzzy project network-II

Table II : Defuzzified values of slack time for fuzzy project -II

Activity	Fuzzy activity time	Slack time (a ,b,c,d)	Original Defuzzified value $\frac{a+b+c+d}{4}$	Proposed Defuzzified value using Eq.(4)
1-2	(2,2,3,4)	(-7,0,6,14)	3.25	3.296
1-3	(2,3,3,6)	(-12,-3,3,12)	0	0
1-5	(2,3,4,5)	(2,6,10,18)	9	9.2
2-4	(2,2,4,5)	(-7,0,6,14)	3.25	3.296
2-5	(2,4,5,8)	(-5,2,7,15)	5	5.10
3-4	(1,1,2,2)	(-6,2,6,15)	4.25	4.306
3-6	(7,8,11,15)	(-12,-3,3,12)	0	0
4-5	(2,3,3,5)	(-7,0,6,14)	3.25	3.296
4-6	(3,3,4,6)	(-6,0,7,14)	3.75	3.790
5-6	(1,1,1,2)	(-7,0,6,14)	3.25	3.296

The paths in fuzzy project network-II are 1-2-4-5-6 ,1-2-4-6 ,1-2-5-6 , 1-3-4-5-6 , 1-3-4-6 ,1-3-6 , and 1-5-6 . The Critical path for fuzzy project network-II is 1-3-6.

Conclusion

A new analytical method for finding critical path in a fuzzy project network has been proposed. We have used a new defuzzification formula for trapezoidal fuzzy number and applied to the float time for each activity in the fuzzy project network to find the critical path. We have shown examples to compare our proposed method with the existing method. The comparison reveal that the method proposed in this paper has shown more effective in determining the activity criticalities and finding the critical path.

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