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Certain Connections on an Almost Unified Para-Norden Contact Metric Manifold

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Abstract

In this paper, we have considered several affine connexions on an almost unified para-norden contact metric manifold. Certain theorems also have been proved which are of great geometrical importance.

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1 Introduction

We consider a differentiable manifold V_n of differentiability class C^{∞}. Let there exist in V_n a tensor F of the type $(1,1)$, a vector field U, a 1-$form$ u and a Riemannian metric g satisfying

$$\overline{\overline{X}} = \lambda^2 X - u(X)U \tag{1.1}$$

$$\overline{U} = 0 \tag{1.2}$$

$$g(\overline{X},\overline{Y}) = \lambda^2 g(X,Y) - u(X)u(Y) \tag{1.3}$$

where

$$F(X) \overset{def}{=} \overline{X}$$

and λ is a complex constant.

Then the set (F, U, u, g) satisfying (1.1) to (1.3) is called an almost unified para-norden contact metric structure and V_n equipped almost unified para-norden contact metric structure is called an almost unified para-norden contact metric manifold.[4]

Remark 1.1. *An almost unified para-norden contact metric manifold is an almost para contact metric manifold [5] or an almost norden contact metric manifold [6] according as $\lambda = \pm 1$ or $\lambda = \pm i$ respectively.*

Agreement 1.1. *All the equations which follow will hold for arbitrary vector fields X, Y, Z, $\ldots$ etc.*

Replacing X by U in (1.1) and using (1.2), we get

$$u(U) = \lambda^2 \tag{1.4}$$

Replacing Y by U in (1.3) and using (1.2) and (1.4), we get

$$g(X,U) \overset{def}{=} u(X) \tag{1.5}$$

Replacing X by $\overline{X}$ in (1.1), we get

$$\overline{\overline{\overline{X}}} = \lambda^2 \overline{X} - u(\overline{X})U \tag{1.6a}$$

Operating F in (1.1) and using (1.2), we get

$$\overline{\overline{\overline{X}}} = \lambda^2 \overline{X} \tag{1.6b}$$

From (1.6a) and (1.6b), we get

$$u(\overline{X}) = 0 \tag{1.7}$$

Agreement 1.2. *An almost unified para-norden contact metric manifold, will, always be denoted by V_n.*

Definition 1.1. *A vector valued, skew-symmetric, bilinear function defined by*

$$S(X,Y) \overset{def}{=} D_X Y - D_Y X - [X,Y] \tag{1.8}$$

is called torsion tensor of a connexion D in a C^∞-manifold V_n.

Definition 1.2. *The tensor K of the type $(1,3)$ defined by*

$$K(X,Y,Z) \overset{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{1.9}$$

is called curvature tensor of the connexion D.

2 Affine Connexion D

We consider in V_n an F-connexion D defined by (*Duggal* 1971, *Mishra* 1984) satisfying

$$(D_X F)Y = 0 \tag{2.1a}$$

From (1.7) and (2.1a) yields

$$D_X \overline{Y} = \overline{D_X Y} \tag{2.1b}$$

Barring Y in (2.1b) and using (1.1) and (2.1b), we get

$$(D_X u)(Y)U + u(Y)(D_X U) = 0 \tag{2.1c}$$

Theorem 2.1. *In V_n, we have*

$$u(Y)u(D_X U) = -\lambda^2 (D_X u)(Y), \tag{2.2}$$

$$\lambda^2 (D_X U) - u(D_X U)U = 0 \tag{2.3}$$

Proof. Operating u in (2.1c) and using (1.4), we get (2.2).
From (1.5), we have

$$(D_X u)U = -u(D_X U) \tag{2.4}$$

Putting U for Y in (2.1c) and using (1.4) and (2.4), we get (2.3). ∎

Theorem 2.2. *In V_n, we have*

$$\begin{aligned} S(\overline{X},\overline{Y}) + \lambda^2 S(X,Y) - u(S(X,Y))U - \overline{S(\overline{X},Y)} - \overline{S(X,\overline{Y})} = -[\overline{X},\overline{Y}] \\ -\lambda^2[X,Y] + u([X,Y])U - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]} \end{aligned} \tag{2.5}$$

where S is the torsion tensor of connexion D.

Proof. From (2.1b), we get

$$\begin{aligned} D_{\overline{X}}\overline{Y} = \overline{D_{\overline{X}}Y}, D_{\overline{Y}}\overline{X} = \overline{D_{\overline{Y}}X} \\ \overline{D_X\overline{Y}} = \overline{\overline{D_X Y}}, \overline{D_Y\overline{X}} = \overline{\overline{D_Y X}} \end{aligned} \tag{2.6}$$

From (1.8), we have

$$\begin{aligned} S(\overline{X},\overline{Y}) + \overline{\overline{S(X,Y)}} - \overline{S(\overline{X},Y)} - \overline{S(X,\overline{Y})} = D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X} + \overline{\overline{D_X Y}} - \overline{\overline{D_Y X}} \\ -\overline{D_{\overline{X}}Y} + \overline{D_Y\overline{X}} - \overline{D_X\overline{Y}} + \overline{D_{\overline{Y}}X} - [\overline{X},\overline{Y}] - \overline{\overline{(X,Y)}} + \overline{[\overline{X},Y]} + \overline{[X,\overline{Y}]} \end{aligned}$$

Using (1.1) and (2.6) in the above equation we get (2.5). ∎

Now, we consider in V_n a scalar valued bilinear function μ, vector valued linear function γ and a 1-$form$ σ given by ($Mishra$ 1973)

$$\mu(X,Y) \overset{def}{=} (D_Y u)(\overline{X}) - (D_X u)(\overline{Y}) + (D_{\overline{Y}} u)(X) - (D_{\overline{X}} u)(Y), \tag{2.7}$$

$$\gamma(X) \overset{def}{=} (D_U F)(X) - (D_X F)(U) - (D_{\overline{X}} U), \tag{2.8}$$

$$\sigma(X) \overset{def}{=} (D_X u)(U) - (D_U u)(X) \tag{2.9}$$

Theorem 2.3. *In V_n, we have*

$$\lambda^2 \mu(X,Y) = -u(X)u(D_{\overline{Y}} U) + u(Y)u(D_{\overline{X}} U), \tag{2.10a}$$

$$\lambda^2 \mu(X,Y) = u(X)u(\gamma(Y)) - u(Y)u(\gamma(X)), \tag{2.10b}$$

$$\lambda^2 \mu(X,Y) = u(X)\sigma(\overline{Y}) - u(Y)\sigma(\overline{X}) \tag{2.10c}$$

Proof. Replacing Y by $\overline{Y}$ in (2.2) and using (1.7), we get

$$(D_X u)(\overline{Y}) = 0 \tag{2.11}$$

Using (2.11) in (2.7), we get

$$\mu(X,Y) = (D_{\overline{Y}} u)(X) - (D_{\overline{X}} u)(Y)$$

Using (2.2) in the above equation we get (2.10a).
Using (2.1a) in (2.8), we get

$$\gamma(X) = -(D_{\overline{X}} U)$$

Which implies

$$u(\gamma(X)) = -u(D_{\overline{X}} U) \tag{2.12}$$

Using (2.12) in (2.10a), we get (2.10b).
Replacing X by $\overline{X}$ in (2.9) and using (2.4) and (2.11), we get

$$\sigma(\overline{X}) = (D_{\overline{X}} u)(U) = -u(D_{\overline{X}} U) \tag{2.13a}$$

From (2.12) and (2.13a), we get

$$\sigma(\overline{X}) = u(\gamma(X)) \tag{2.13b}$$

Using (2.13b) in (2.10b), we get (2.10c). ∎

Theorem 2.4. *In V_n, we have*

$$\lambda^2\mu(\overline{X},Y) = \lambda^2 u(Y)u(D_X U) - u(Y)u(X)u(D_U U), \tag{2.14a}$$

$$\lambda^2\mu(X,\overline{Y}) = -\lambda^2 u(X)u(D_Y U) + u(X)u(Y)u(D_U U), \tag{2.14b}$$

$$\lambda^2\mu(\overline{X},Y) + \lambda^2\mu(X,\overline{Y}) = -u(Y)u(\gamma(\overline{X})) + u(X)u(\gamma(\overline{Y})) \tag{2.14c}$$

Proof. Barring X and Y separatly in (2.10a) and using (1.1) and (1.7), we get (2.14a) and (2.14b) respectively.

Barring X and Y separatly in (2.10b) and using (1.7), we get

$$\lambda^2\mu(\overline{X},Y) = -u(Y)u(\gamma(\overline{X})), \tag{2.15a}$$

$$\lambda^2\mu(X,\overline{Y}) = u(X)u(\gamma(\overline{Y})) \tag{2.15b}$$

Adding (2.15a) and (2.15b), we get (2.14c). ∎

Corollary 2.4.1. *In V_n, we have*

$$\mu(\overline{X},Y) + \mu(X,\overline{Y}) = -u(Y)\sigma(X) + u(X)\sigma(Y) \tag{2.16}$$

Proof. From (2.9) implies

$$\sigma(U) = 0$$

Barring X in (2.13b) and using (1.1) and the above equation, we get

$$\lambda^2\sigma(X) = u(\gamma(\overline{X})) \tag{2.17}$$

Using (2.17) in (2.14c), we get (2.16). ∎

Theorem 2.5. *In V_n, we have*

$$\overline{K(X,Y,\overline{Z})} = \lambda^2 K(X,Y,Z) - u(K(X,Y,Z))U \tag{2.18}$$

Proof. Replacing Z by $\overline{Z}$ in (1.9) and using (2.1b), we get

$$K(X,Y,\overline{Z}) = \overline{K(X,Y,Z)} \tag{2.19}$$

Barring (2.19) and using (1.1), we get (2.18). ∎

Corollary 2.5.1. *In V_n, we have*

$$\lambda^2 K(X,Y,U) = u(K(X,Y,U))U, \tag{2.20a}$$

$$\lambda^2 u(K(X,Y,Z)) - u(Z)u(K(X,Y,U)) = 0 \tag{2.20b}$$

Proof. Replacing Z by U in (2.19) and using (1.2), we get

$$\overline{K(X,Y,U)} = 0 \tag{2.21a}$$

Barring (2.21a) and using (1.1), we get (2.20a).

Operating u on (2.19) and using (1.7), we get

$$u(K(X,Y,\overline{Z})) = 0 \tag{2.21b}$$

Barring Z in (2.21b) and using (1.1), we get (2.20b). ∎

Theorem 2.6. *If an Affine connexion D is symmetric in V_n, we have*

$$\begin{aligned}\lambda^2(K(\overline{X},\overline{Y},Z)+K(\overline{Y},\overline{Z},X)+K(\overline{Z},\overline{X},Y)) &= u(Z)K(\overline{X},\overline{Y},U)\\ &+u(X)K(\overline{Y},\overline{Z},U)+u(Y)K(\overline{Z},\overline{X},U)\end{aligned} \tag{2.22}$$

Proof. From (2.19) yields

$$K(X,Y,\overline{Z})+K(Y,Z,\overline{X})+K(Z,X,\overline{Y}) = \overline{K(X,Y,Z)}+\overline{K(Y,Z,X)}+\overline{K(Z,X,Y)}$$

Using in above the Bianchi's first Identity satisfied by a curvature tensor for symmetric connexion D, we get

$$K(X,Y,\overline{Z})+K(Y,Z,\overline{X})+K(Z,X,\overline{Y}) = 0 \tag{2.23}$$

Barring X, Y, Z in (2.23) and using (1.1), we get (2.22). ∎

3 Affine Connexion $D^{'}$

let us consider in V_n an Affine connexion $D^{'}$ satisfying

$$u(Y)(D^{'}_X U) + (D^{'}_X u)(Y)U = 0 \tag{3.1}$$

Theorem 3.1. *In V_n, we have*

$$\lambda^2(D^{'}_X U) = u(D^{'}_X U)U \tag{3.2}$$

Proof. Barring (3.1) and using (1.2), we get

$$\overline{(D^{'}_X U)} = 0 \tag{3.3}$$

Barring (3.3) and using (1.1), we get (3.2). ∎

Theorem 3.2. *In V_n, we have*

$$\lambda^2 divU = u(D^{'}_U U) \tag{3.4a}$$

where

$$divX \overset{def}{=} (C^1_1 \nabla X), \tag{3.4b}$$

$$(\nabla X)(Y) \overset{def}{=} D^{'}_Y X \tag{3.4c}$$

Proof. Contracting (3.1) with respect to X and using (3.4b) and (3.4c), we get

$$u(Y)divU + (D'_U u)(Y) = 0 \tag{3.5}$$

Replacing U by Y in (3.5) and using (1.4) and (2.4), we get (3.4a). ∎

Theorem 3.3. *In V_n, we have*

$$u(Y)u(D'_X U) = -\lambda^2 (D'_X u)(Y), \tag{3.6a}$$

$$\lambda^2 (D'_X u)(Y) u(D'_Z U) = u(Y) u(D'_Z U)(D'_X u)U \tag{3.6b}$$

Proof. Operating by u on (3.1) and using (1.4), we get (3.6a).
Multyplying (3.6a) by $u(D'_Z U)$, we get

$$u(Y)u(D'_X U)u(D'_Z U) = -\lambda^2 (D'_X u)(Y) u(D'_Z U) \tag{3.7}$$

Using (2.4) in (3.7), we get (3.6b). ∎

4 Affine Connexion D^o

let us consider in V_n an Affine connexion D^o satisfying

$$u(Y)(D^o_X U) + (D^o_X u)(Y)U = 0, \tag{4.1a}$$

$$(D^o_X F)(Y) + (D^o_Y F)(X) = 0 \tag{4.1b}$$

Theorem 4.1. *In V_n, we have*

$$\overline{D^o_X \overline{\overline{Y}}} + \overline{D^o_Y \overline{\overline{X}}} - \lambda^2 (D^o_X Y + D^o_Y X) = -(u(D^o_X Y)U + u(D^o_Y X)U), \tag{4.2}$$

$$D^o_{\overline{Y}} \overline{X} - \lambda^2 (D^o_Y X) = \overline{D^o_{\overline{Y}} X} - \overline{D^o_Y \overline{\overline{X}}} - u(D^o_Y X)U, \tag{4.3}$$

$$\overline{D^o_{\overline{Y}} \overline{\overline{X}}} - \lambda^2 \overline{(D^o_Y X)} + \lambda^2 (D^o_Y \overline{X}) = u(D^o_Y \overline{X})U - u(D^o_{\overline{Y}} X)U + \lambda^2 (D^o_{\overline{Y}} X), \tag{4.4}$$

$$\overline{D^o_U \overline{\overline{Y}}} - \lambda^2 (D^o_U Y) = -u(D^o_U Y)U \tag{4.5}$$

Proof. Since

$$F(X) \overset{def}{=} \overline{X}$$

Using the above equation in (4.1b), we get

$$D^o_X \overline{Y} + D^o_Y \overline{X} = \overline{D^o_X Y} + \overline{D^o_Y X} \tag{4.6}$$

Barring (4.6) and using (1.1), we get (4.2).
Barring Y in (4.6) and using (1.1), (4.1a) and (4.2), we get (4.3).
Barring (4.3) and using (1.1), (1.2) and (1.4), we get (4.4).
Replacing X by U in (4.2) and using (1.2) and (3.2), we get (4.5). ∎

5 Affine Connexion D^*

let us consider in V_n an Affine connexion D^* satisfying

$$u(Y)(D^*_X U) + (D^*_X u)(Y)U = 0, \tag{5.1a}$$

$$(D^*_X F)(Y) + (D^*_{\overline{X}} F)(\overline{Y}) = 0 \tag{5.1b}$$

Theorem 5.1. *In V_n, we have*

$$D^*_X \overline{Y} - u(D^*_{\overline{X}} Y)U = \overline{D^*_X Y} + \overline{D^*_{\overline{X}} \overline{Y}} - \lambda^2 D^*_{\overline{X}} Y, \tag{5.2a}$$

$$(D^*_X F)(Y) + \overline{(D^*_{\overline{X}} F)(Y)} = 0, \tag{5.2b}$$

$$\overline{(D^*_X F)(Y)} + \lambda^2 (D^*_{\overline{X}} F)(Y) = 0 \tag{5.2c}$$

Proof. Since

$$F(X) \overset{def}{=} \overline{X}$$

Using the above equation in (5.1b), we get

$$D^*_X \overline{Y} + D^*_{\overline{X}} \overline{\overline{Y}} = \overline{D^*_X Y} + \overline{D^*_{\overline{X}} \overline{Y}} \tag{5.3}$$

Using (1.1) in (5.3), we get (5.2a).
Using (2.1b) in (5.1b), we get (5.2b).
Barring Y in (5.1a) and using (1.7), we get

$$(D^*_X u)(\overline{Y}) = 0 = u(D^*_X F)(Y) \tag{5.4}$$

Barring (5.2b) and using (1.1) and (5.4), we get (5.2c). ∎

Theorem 5.2. *In V_n, we have*

$$\overline{D^*_U \overline{\overline{Y}}} - \lambda^2 D^*_U Y + u(D^*_U Y)U = 0, \tag{5.5}$$

$$D^*_{\overline{X}} \overline{Y} - \lambda^2 \overline{D^*_X \overline{\overline{Y}}} + \lambda^4 D^*_X Y = \overline{D^*_{\overline{X}} Y} + \lambda^2 u(D^*_X Y)U \tag{5.6}$$

Proof. Replacing X by U in (5.3) and using (1.4), we get

$$D^*_U \overline{Y} = \overline{D^*_U Y} \tag{5.7}$$

Barring (5.7) and using (1.1), we get (5.5).

Barring X in (5.2a) and using (1.1) and (5.5), we get (5.6). ∎

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