A New Method for Finding an Optimal More-For-Less Solution of Transportation Problems with Mixed Constraints

P. Pandian and G. Natarajan

Department of Mathematics, School of Advanced Sciences
VIT University, Vellore-14, Tamil Nadu, India
pandian61@rediffmail.com

Abstract

A new method called zero point method is proposed for finding an optimal solution for transportation problems with mixed constraints in a single stage. Using the zero point method, we propose a new method for finding an optimal more-for-less solution for transportation problems with mixed constraints. The optimal more-for-less solution procedure is illustrated with numerical examples.

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1 Introduction

The transportation problem (TP) is a special class of linear programming problem, which deals with shipping commodities from sources to destinations. The objective of the TP is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. The TP finds application in industry, planning, communication network, scheduling, transportation and allotment etc. In literature, a good amount of research [2,6] is available to obtain an optimal solution for TPs with equality constraints. In real life, most of the TPs have mixed constraints, accommodating many applications not only in the distribution problems but also, in job scheduling, production inventory and investment analysis. The TPs with mixed constraints are not addressed in the literature because of the
rigour required to solve these problems optimally. A literature search revealed no systematic method for finding an optimal solution for TPs with mixed constraints.

The more-for-less (MFL) paradox in a TP occurs when it is possible to ship more ‘total goods’ for less (or equal) ‘total cost’ while shipping the same amount or more from each origin and to each destination, keeping all shipping costs non-negative. The occurrence of MFL in distribution problems is observed in nature. The existing literature [1, 3, 4, 5, 9-15] has demonstrated the identifying cases where MFL paradoxical situation exists and also, has provided various methods for finding MFL solution for transportation problems. Gupta et al.[4] and Arsham [1] obtained the more-for-less solution for the TPs with mixed constraints by relaxing the constraints and by introducing new slack variables. While yielding the best more-for-less solution, their method is tedious since it introduces more variables and requires solving sets of complex equations. The perturbed method was used for solving the TPs with constraints in [7,5,8]. Adlakha et al.[14] proposed a heuristic method for solving TPs with mixed constraints which is based on the theory of shadow price. In the heuristic algorithm for an MFL solution in Adlakha et al.[14], Vogel Approximation Method (VAM) and MODI (Modified Distribution) method were used. The primary goal of the MFL method is to minimize the total cost and not merely to maximize the shipment load transported.

In this paper, we develop a new method called zero point method for finding an optimal solution of TPs with mixed constraints in a single stage. Using the zero point method, we propose a new method for finding an optimal MFL solution for TPs with mixed constraints. The optimal MFL solution procedure is illustrated with the help of numerical examples. The two proposed methods are very simple, easy to understand and apply. The MFL situation exists in reality and it could present managers with an opportunity for shipping more units for less or the same cost. The more-for-less analysis could be useful for managers in making important decisions such as increasing warehouse/plant capacity, or advertising efforts to increase demand at certain markets.

2 Transportation problem with mixed constraints

Consider the mathematical model for a TP with mixed constraints.

\begin{align*}
\text{(P)} & \quad \text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} & \quad \sum_{i=1}^{m} x_{ij} \geq a_j, \quad j \in Q \quad (1) \\
& \quad \sum_{i=1}^{m} x_{ij} \leq a_j, \quad j \in T \quad (2)
\end{align*}
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\[
\sum_{i=1}^{m} x_{ij} = a_j, \quad j \in S
\]  
\[
\sum_{j=1}^{n} x_{ij} \geq b_i, \quad i \in U
\]
\[
\sum_{j=1}^{n} x_{ij} \leq b_i, \quad i \in V
\]
\[
\sum_{j=1}^{n} x_{ij} = b_i, \quad i \in W
\]
\[
x_{ij} \geq 0, \quad i = 1,2,...,m \text{ and } j = 1,2,...,n \text{ and integers}
\]

where

\( Q, T, \) and \( S \) are pairwise disjoint subsets of \( \{1,2,3,...,n\} \) such that \( Q \cup T \cup S = \{1,2,3,...,n\} \);

\( U, V, \) and \( W \) are pairwise disjoint subsets \( \{1,2,3,...,m\} \) such that \( U \cup V \cup W = \{1,2,3,...,m\} \);

\( c_{ij} \) is the cost of shipping one unit from supply point \( i \) to the demand point \( j \);

\( b_i \) is the supply at supply point \( i \);

\( a_j \) is the demand at demand point \( j \) and

\( x_{ij} \) is the number of units shipped from supply point \( i \) to demand point \( j \).

Now, we follow the rules given below for finding the maximum possible allotment to a cell in the transportation table whose corresponding demand and supply limits are known:

(i) The maximum possible allotment to the cell whose limits pair is \( \{ \leq a,= b \} \),

\[
= \begin{cases} 
  b : a \geq b \\
  a : a < b
\end{cases}
\]

(ii) The maximum possible allotment to the cell whose limits pair is \( \{ \geq a,= b \} \), \( = b \).

(iii) The maximum possible allotment to the cell whose limits pair is \( \{ \leq a,\leq b \} \), \( = 0 \).

(iv) The maximum possible allotment to the cell whose limits pair is \( \{ = a,= b \} \),

\[
= \text{minimum of } \{ a,b \}
\]

(v) The maximum possible allotment to the cell whose limits pair is \( \{ \geq a,\geq b \} \),

\[
= \text{maximum of } \{ a,b \} \text{ and}
\]
(vi) The maximum possible allotment to the cell whose limits pair is \( \{ a, \leq b \} \),
\[
= \begin{cases} 
  b : a \geq b \\
  a : a < b
\end{cases}
\]

3 Zero point method

We now introduce a new method called the zero point method for finding an optimal solution to a transportation problem with mixed constraints in a single stage.

The zero point method proceeds as follows.

Step 1. Construct the transportation table for the given TP with mixed constraints.

Step 2. Subtract each row entries of the transportation table from the row minimum and then subtract each column entries of the resulting transportation table after using the Step 1 from the column minimum.

Step 3. Check if each column demand can be accomplished from the joint of row supplies whose reduced costs in that column are zero. Also, check if each row supply can be accomplished from the joint of column demands whose reduced costs in that row are zero. If so, go to Step 6. (Such reduced transportation table is called the allotment table). If not, go to Step 4.

Step 4. Draw the minimum number of horizontal lines and vertical lines to cover all the zeros of the reduced transportation table such that some entries of row(s) or/and column(s) which do not satisfy the condition of the Step 3 are not covered.

Step 5: Develop the new revised reduced transportation table as follows:

(i) Find the smallest entry of the reduced cost matrix not covered by any lines.

(ii) Subtract this entry from all the uncovered entries and add the same to all entries lying at the intersection of any two lines.

and then, go to Step 3.

Step 6: Select a cell in the reduced transportation table whose reduced cost is the maximum cost. Say \((\alpha, \beta)\). If there is more than one, then select any one.

Step 7: Select a cell in the \(\alpha\)-row or/and \(\beta\)-column of the reduced transportation table which is the only cell whose reduced cost is zero and then, allot the maximum possible to that cell such that its row or its column condition is satisfied. If such cell does not occur for the maximum value,
find the next maximum so that such a cell occurs. (If such a cell does not occur, select a zero reduced cell in the allotment table whose original cost is the least).

Step 8: Reform the reduced transportation table after deleting the fully used supply points and the fully achieved demand points and also modify it to include the not fully used supply points and the not fully achieved demand points.

Step 9: Repeat Step 6 to Step 8 until all supply points are fully used and all demand points are fully achieved.

Step 10: This allotment yields a solution to the given TP with mixed constraints.

Remark 1: If the given TP has equality constraints, write the given TP into a balanced TP one (if it is not so) and then, apply the zero point method for finding an optimal solution with the following modified Step 3:

Step 3. Check if each column demand is less than the sum of row supplies whose reduced costs in that column are zero. Also, check if each row supply is less than the sum of column demands whose reduced costs in that row are zero. If so, go to Step 6. (Such reduced transportation table is called the allotment table). If not, go to Step 4.

Now, we prove that the solution to a TP with mixed constraints obtained by the zero point method is an optimal solution to the problem.

Theorem 1. Any optimal solution to the problem \((P_1)\) where

\[(P_1) \text{ Minimize } z^* = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)x_{ij} \]

subject to (1) to (7) are satisfied, where \(u_i\) and \(v_j\) are some real values, is an optimal solution to the problem \((P)\) where

\[(P) \text{ Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} \]

subject to (1) to (7) are satisfied.

Proof. Now, 

\[z^* = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} u_i x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} v_j x_{ij} = z - \sum_{i=1}^{m} u_i b_i - \sum_{j=1}^{n} v_j a_j. \]

Since \(\sum_{i=1}^{m} u_i b_i\) and \(\sum_{j=1}^{n} v_j a_j\) are independent of \(x_{ij}\), for all \(i\) and \(j\), we can conclude that any optimal solution to the problem \((P_1)\) is also an optimal solution to the problem \((P)\).
Hence the theorem.

**Theorem 2.** If \( \{ x^*_{ij}, i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n \} \) is a feasible solution to the problem (P) and \( c_{ij} - u_i - v_j \geq 0 \), for all \( i \) and \( j \) where \( u_i \) and \( v_j \) are some real values, such that the minimum of \( \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)x_{ij} \) subject to (1) to (7) are satisfied, is zero, then \( \{ x^*_{ij}, i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n \} \) is an optimum solution to the problem (P).

**Proof.** From the Theorem 1., the result follows.

**Theorem 3.** The solution obtained by the zero point method for a TP with mixed constraints (P) is an optimal solution for the problem (P).

**Proof.** We, now describe the zero point method in detail.

We construct the transportation table \([c_{ij}]\) for the given TP.

Let \( u_i \) be the minimum of \( i \)-the row of the table \([c_{ij}]\).

Now, we subtract \( u_i \) from the \( i \)-th row entries so that the result table is \([c_{ij} - u_i]\).

Let \( v_j \) be the minimum of \( j \)-th column of the resulting table \([c_{ij} - u_i]\).

Now, we subtract \( v_j \) from the \( j \)-th column entries so that the resulting table is \([c_{ij} - u_i - v_j]\). It may be noted that \( c_{ij} - u_i - v_j \geq 0 \), for all \( i \) and \( j \) and each row and each column of the resulting table \([c_{ij} - u_i - v_j]\) has at least one zero entry.

Each column demand condition of the resulting table \([c_{ij} - u_i - v_j]\) is accomplished from the joint of supplies whose reduced costs in the column are zero. Further, each row supply condition of the resulting table \([c_{ij} - u_i - v_j]\) is accomplished the joint of column demands whose reduced costs in the row are zero (If not so, as per direction given in the Step 4 and 5 in the zero point method, we can make it that). The current resulting table is the allotment table.

We find a cell in the allotment table \([c_{ij} - u_i - v_j]\) whose reduced cost is the highest. Say \((\alpha, \beta)\). We allot the maximum possible to a cell in the \( \alpha \)-row or/and \( \beta \)-column in accordance with the direction given in the Step 7. The resulting transportation table is reformed after deleting the fully used supply points and the fully achieved demand points. Also, the supply points not fully used and the demand points not fully achieved are modified. We repeat this procedure till the total supply is fully used and the total demand is fully achieved.

Finally, we have a solution \( \{ x_{ij}, i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n \} \) for the reduced TP with mixed constraints whose cost matrix is \((c_{ij} - u_i - v_j)\) such that
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\[ x_{ij} = 0 \text{ for } c_{ij} - u_i - v_j \geq 0 \text{ and } x_{ij} > 0 \text{ for } c_{ij} - u_i - v_j = 0. \]

Therefore, the minimum of
\[ \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)x_{ij} \]
subject to (1) to (7) are satisfied, is zero. Thus, by the Theorem 2., the solution \{ \( x_{ij}, i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n \) \} is an optimal solution of the problem (P).

Hence the theorem.

4 More-for-less solution

Asharm [1] proved that the existence of a MFL situation in a regular TP requires only one condition namely, the existence of a location with negative plant-to-market shipping shadow price. The shadow prices are easily calculated from the optimal solution of the TP with mixed constraints. The MFL solution is obtained from the optimal solution distribution by increasing and decreasing the shipping quantities while maintaining the minimum requirements for both supply and demand. The plant-to-market shipping shadow price (also called Modi index) at a cell \((i,j)\) is \( u_i + v_j \) where \( u_i \) and \( v_j \) are shadow prices corresponding to the cell \((i,j)\). The negative Modi index at a cell \((i,j)\) indicates that we can increase the ith plant capacity / the demand of the jth market at the maximum possible level.

Theorem 4. The optimal MFL solution of a TP with mixed constraints is an optimal solution of a TP with mixed constraints by changing the sign of columns and rows having negative Modi indices from \( \leq \) to \( = \) and \( = \) to \( \geq \).

Proof: Since the existence of a MFL situation in a TP requires only one condition namely, the existence of a location with negative Modi index. Therefore, the negative Modi index at a cell \((i,j)\), \( u_i + v_j \) indicates that we can achieve the supply of the ith source / the demand of the jth destination at the maximum possible level. Construct a TP with mixed constraints obtained from the given problem by changing the sign of columns and rows having negative Modi indices from \( \leq \) to \( = \) and \( = \) to \( \geq \) in the given problem. The newly constructed TP with mixed constraints is a TP problem with mixed constraints such that all the columns and rows having negative Modi indices can be achieved at the maximum level. Therefore, any solution of the newly constructed TP with mixed constraints is an MFL solution to the given problem. Thus, the optimal solution of the newly constructed TP with mixed constraints is an optimal MFL solution to the given TP with mixed constraints.

Hence the theorem.

Optimal MFL procedure:

We use the following procedure for finding an optimal MFL solution to a TP with mixed constraints.
Step 1: Prepare the Modi index matrix for the optimal solution of the TP with mixed constraints obtained by the zero point method.

Step 2: Identify the negative Modi indices and related columns and rows. If none exists, this is an optimal solution to the TP with mixed constraints (no MFL paradox is present). STOP.

Step 3: Form a new TP with mixed constraints by changing the sign of columns and rows having negative Modi indices from ≤ to = and = to ≥ in the given problem.

Step 4: Solve the TP with mixed constraints obtained from the Step 3 using the zero point method.

Step 5: The optimal solution for the TP with mixed constraints obtained from the Step 4 is an optimal MFL solution of the given TP with mixed constraints (by the Theorem 4.).

5 Numerical example

The proposed method for finding an optimal MFL solution to a TP with mixed constraints is illustrated by the following example.

Example 1. Consider the following TP with mixed constraints.

We obtain the following allotment table for the given problem by using the Step 1 to the Step 5 of the zero point method.

Then, we obtain the following allotment for the given problem by using allotment rules of the zero point method.
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Therefore, the optimal solution for the given problem is \( x_{11} = 5 \), \( x_{21} = 3 \), \( x_{22} = 10 \), \( x_{23} = 5 \) and \( x_{33} = 0 \) for a flow of 18 units with the minimum total transportation cost is $63.

Now, the Modi index matrix for the optimal solution of given problem is given below.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
v_1 & 2 & -1 & -3 \\
u_1 & 2 \\
v_2 & 6 & 3 & 1 \\
u_2 & 6 \\
v_3 & 7 & 4 & 2 \\
u_3 & 7 \\
v_j & 0 & -3 & -5 \\
\end{array}
\]

Since the first row and the second and third columns have negative Modi indices, we consider the following new TP with mixed constraints.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
\text{Supply} & 5 & 6 & 9 \\
\geq & \geq & \leq \\
1 & 2 & 3 \\
\hline
\text{Demand} & 8 & 10 & 5 \\
\geq & \geq & = \\
\end{array}
\]

Next, we obtain the following allotment table for the new TP with mixed constraints by using the Step 1 to the Step 5 of the zero point algorithm.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
\text{Supply} & 5 & 6 & 7 \\
\geq & \geq & \leq \\
1 & 2 & 3 \\
\hline
\text{Demand} & 8 & 10 & 5 \\
\geq & \geq & = \\
\end{array}
\]

Now, we obtain the following allotment for the new TP with mixed constraints by using allotment rules of the zero point method.
Now, the Modi index matrix for the solution of the new TP is given below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>0</td>
<td></td>
<td>≥ 5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>5</td>
<td></td>
<td>≥ 6</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0</td>
<td></td>
<td>≤ 9</td>
</tr>
</tbody>
</table>

Demand: ≥ 8 ≥ 10 = 5

Now, the Modi index matrix for the solution of the new TP is given below.

\[
\begin{array}{ccc}
  & v_1 & v_2 & v_3 & u_i \\
 u_1 & 2   & 5   & 3   & 2 \\
u_2 & 0   & 3   & 1   & 0 \\
u_3 & 1   & 4   & 2   & 1 \\
 v_j & 0   & 3   & 1   & \\
\end{array}
\]

Since all the Modi indices are positive, the current solution is an optimal solution of the new TP with mixed constraints. Thus, by Theorem 1., the optimal MFL solution for the given TP with mixed constraints is \( x_{11} = 8, \ x_{12} = 0, \ x_{22} = 10, \ x_{23} = 5, \) and \( x_{33} = 0 \) for a flow of 23 units with the total transportation cost is $51.

The solution is better than the solution obtained by [4,14] because the shipping rate per unit is 2.22.

Note 1: For calculating Modi indices, we need \( n + m - 1 \) loading cells. So, we keep the cells that would be loaded using the zero point method even with a load of zero.

6 Conclusion

We have provided a method called zero point method to find an optimal solution of TPs with mixed constraints in single stage. We have attempted to develop a new process based on the zero point method to find an optimal MFL solution to TPs with mixed constraints. The proposed method for an optimal MFL solution is very simple, easy to understand and apply. The MFL analysis could be useful for mangers in making strategic decisions such as increasing a ware-house stocking level or plant production capacity and advertising efforts to increase demand at certain markets. So, the new method for an optimal MFL solution using zero point method can serve managers by providing one of the best MFL solutions to a variety of distribution problems with mixed constraints.

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