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Some Properties of Quarter-Symmetric Non-Metric Connection in a Kähler Manifold

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Abstract

In the present paper, we studied the properties of a quarter-symmetric non-metric connection in a Kähler manifold.

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1 Introduction

Golab [1] defined and studied the properties of quarter-symmetric connection on a differentiable manifold. In this series Mishra and Pandey [2] defined quarter-symmetric metric $F-$ connections and studied some of its properties. In 2003, Sengupta and Biswas defined a quarter-symmetric non-metric connection on a Sasakian manifold [3]. Recently, Chaubey and Ojha [4] defined a quarter-symmetric non-metric connection on an almost Hermite manifold. They also have been studied the properties of a quarter-symmetric metric connection in an Einstein manifold [5]. The present paper deals with the different geometrical properties of a Kähler manifold equipped with the quarter-symmetric non-metric connection [4].

2 Preliminaries

If on an even dimensional differentiable manifold $V_n, n = 2m$, of differentiability class C^{r+1}, there exists a vector valued function F of differentiability class C^r, satisfying

$$(a) \quad \overline{\overline{X}} + X = 0 \quad and \quad (b) \quad \overline{X} \overset{\text{def}}{=} FX, \tag{1}$$

for arbitrary vector field X, then V_n is said to be an almost complex manifold and $\{F\}$ is said to give an almost complex structure to V_n [Ehresmann, 1947,1950], [6]. If g denotes the Hermitian metric such that

$$g(\overline{X}, \overline{Y}) = g(X, Y), \tag{2}$$

for arbitrary vector fields X and Y, then an almost complex manifold endowed with an almost complex structure F, know as an almost Hermite manifold [6].

If we define

$$'F(X, Y) \overset{\text{def}}{=} g(\overline{X}, Y), \tag{3}$$

then we have

$$'F(X, Y) +' F(Y, X) = 0. \tag{4}$$

An almost Hermite manifold satisfying

$$(D_X{}'F)(Y, Z) = 0, \tag{5}$$

$$(D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) = 0, \tag{6}$$

for arbitrary vector fields X, Y, Z; are respectively called Kähler, almost Kähler manifolds [6].

3 Quarter-symmetric non-metric connection

A linear connection B on (V_n, g) defined as

$$B_X Y = D_X Y + u(Y)FX, \tag{7}$$

for arbitrary vector fields X and Y, is said to be a quarter-symmetric non-metric connection [3], [4] if the torsion tensor S of the connection B and the metric tensor g are given by

$$S(X, Y) = u(Y)FX - u(X)FY \tag{8}$$

and

$$(B_X g)(Y, Z) = -u(Y)g(FX, Z) - u(Z)g(FX, Y), \tag{9}$$

for arbitrary vector fields X, Y, Z; where u is $1-$form on V_n with T as associated vector fields, i.e.,

$$u(X) \overset{\text{def}}{=} g(X,T) \tag{10}$$

and D being the Riemannian connection. If in addition

$$B_X F = 0, \tag{11}$$

then B is known as a quarter-symmetric non-metric $F-$connection.

Theorem 3.1 *An almost Hermite manifold equipped with a quarter-symmetric non-metric $F-$connection B is an almost Kähler manifold.*

Proof We have,

$$X('F(Y,Z)) = (B_X{}'F)(Y,Z) +' F(B_X Y, Z) +' F(Y, B_X Z)$$
$$= (D_X{}'F)(Y,Z) +' F(D_X Y, Z) +' F(Y, D_X Z),$$

then in view of (7), above becomes

$$(B_X{}'F)(Y,Z) = (D_X{}'F)(Y,Z) + u(Y)g(X,Z) - u(Z)g(X,Y) \tag{12}$$

In view of (11), (12) becomes

$$(D_X{}'F)(Y,Z) = u(Z)g(X,Y) - u(Y)g(X,Z) \tag{13}$$

Taking cyclic sum of X, Y, Z in (13), we obtain

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) = 0$$

Hence the theorem.

Theorem 3.2 *An almost Hermite manifold equipped with a quarter-symmetric non-metric $F-$connection is a Hermite manifold.*

Proof Differentiating $FY = \overline{Y}$ with respect to X and using (1) (a), (1) (b) and (7), we get

$$(B_X F)(Y) = (D_X F)(Y) - u(Y)\overline{X} - u(\overline{Y})\overline{X} \tag{14}$$

In consequence of (11), (14) becomes

$$(D_X F)(Y) = u(Y)\overline{X} + u(\overline{Y})\overline{X} \tag{15}$$

The Nijenhuis tensor in an almost Hermite manifold [6] is

$$N(X,Y) = (D_{\overline{X}}F)(Y) - (D_{\overline{Y}}F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)} \tag{16}$$

In view of (1) (a) and (15), (16) becomes

$$N(X,Y) = 0$$

An almost Hermite manifold for which the Nijenhuis tensor vanishes, is called a Hermite manifold [6, pp- 70]. Hence the statement of the theorem.

4 Curvature tensor with respect to the quarter-symmetric non-metric connection

The curvature tensor R with respect to the quarter-symmetric non-metric connection B is defined

$$R(X,Y,Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]}Z$$

which satisfy [3]

$$R(X,Y,Z) = K(X,Y,Z) + (D_X u)(Z)(\overline{Y}) - (D_Y u)(Z)(\overline{X}) + u(Z)\left\{(D_X F)(Y) - (D_Y F)(X)\right\}, \tag{17}$$

where

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \tag{18}$$

is the curvature tensor of the Riemannian connection D.

If we define

$$'R(X,Y,Z,U) \overset{\text{def}}{=} g(R(X,Y,Z),U)$$

and

$$'K(X,Y,Z,U) \overset{\text{def}}{=} g(K(X,Y,Z),U),$$

then in a Kähler manifold (17) becomes

$$'R(X,Y,Z,U) =' K(X,Y,Z,U) + (D_X u)(Z) g(\overline{Y},U) - (D_Y u)(Z) g(\overline{X},U) \tag{19}$$

Theorem 4.1 *If a Kähler manifold admitting a quarter-symmetric non-metric connection B, then $(D_X u)(Z)$ is hybrid if and only if any one of the following holds*

$$(a) \quad 'R(\overline{X},\overline{Y},Z,U) =' R(X,Y,\overline{Z},\overline{U})$$

$$(b) \quad 'R(\overline{X},\overline{Y},\overline{Z},\overline{U}) =' R(X,Y,Z,U)$$

$$(c) \quad 'R(X,\overline{Y},\overline{Z},U) =' R(\overline{X},Y,Z,\overline{U}).$$

Proof Barring Z and U in (19) and using (2), we get

$$'R(X,Y,\overline{Z},\overline{U}) =' K(X,Y,\overline{Z},\overline{U}) + (D_X u)(\overline{Z}) g(Y,U) - (D_Y u)(\overline{Z}) g(X,U) \tag{20}$$

Again barring X and Y in (19) and using (1) (a), we find

$$'R(\overline{X},\overline{Y},Z,U) =' K(\overline{X},\overline{Y},Z,U) - (D_{\overline{X}} u)(Z) g(Y,U) - (D_{\overline{Y}} u)(Z) g(X,U) \tag{21}$$

Subtracting (21) from (20) and using

$$'K(\overline{X},\overline{Y},Z,U) =' K(X,Y,\overline{Z},\overline{U}),$$

we obtain

$$'R(X,Y,\overline{Z},\overline{U}) -' R(\overline{X},\overline{Y},Z,U) = [(D_X u)(\overline{Z}) + (D_{\overline{X}} u)(Z)]g(Y,U) - [(D_Y u)(\overline{Z}) + (D_{\overline{Y}} u)(Z)]g(X,U) \tag{22}$$

We have from (22)

$$'R(\overline{X},\overline{Y},Z,U) =' R(X,Y,\overline{Z},\overline{U})$$

if and only if

$$(D_X u)(\overline{Z}) + (D_{\overline{X}} u)(Z) = 0$$
$$\Leftrightarrow \quad (D_X u)(Z) \text{ is hybrid.}$$

Again barring X, Y, Z and U in (19) and using (1) (a), we obtain

$$'R(\overline{X},\overline{Y},\overline{Z},\overline{U}) =' K(\overline{X},\overline{Y},\overline{Z},\overline{U}) - (D_{\overline{X}} u)(\overline{Z})g(Y,\overline{U}) + (D_{\overline{Y}} u)(\overline{Z})g(X,\overline{U}) \tag{23}$$

Subtracting (19) from (23) and using

$$'K(\overline{X},\overline{Y},\overline{Z},\overline{U}) = K(X,Y,Z,U),$$

we find

$$'R(\overline{X},\overline{Y},\overline{Z},\overline{U}) -' R(X,Y,Z,U) = [(D_X u)(Z) - (D_{\overline{X}} u)(\overline{Z})]g(Y,\overline{U}) + [(D_Y u)(Z) - (D_{\overline{Y}} u)(\overline{Z})]g(\overline{X},U) \tag{24}$$

Since $'R(\overline{X},\overline{Y},\overline{Z},\overline{U}) =' R(X,Y,Z,U)$ if and only if $(D_{\overline{X}} u)(\overline{Z}) = (D_X u)(Z) \Leftrightarrow$ $(D_X u)(Z)$ is hybrid. Also barring Y and Z in (19) and using (1) (a) and (2), we have

$$'R(X,\overline{Y},\overline{Z},U) =' K(X,\overline{Y},\overline{Z},U) - (D_X u)(\overline{Z})g(Y,U) - (D_{\overline{Y}} u)(\overline{Z})g(\overline{X},U) \tag{25}$$

In the last, barring X and U in (19) and using (1) (a) and (2), we get

$$'R(\overline{X},Y,Z,\overline{U}) =' K(\overline{X},Y,Z,\overline{U}) + (D_{\overline{X}} u)(Z)g(Y,U) + (D_Y u)(Z)g(X,\overline{U}) \tag{26}$$

Subtracting (25) from (26) and using

$$'K(X,\overline{Y},\overline{Z},U) =' K(\overline{X},Y,Z,\overline{U}),$$

we have

$$'R(\overline{X},Y,Z,\overline{U}) -' R(X,\overline{Y},\overline{Z},U) = [(D_Y u)(Z) - (D_{\overline{Y}} u)(\overline{Z})]g(\overline{X},U) - [(D_X u)(\overline{Z}) + (D_{\overline{X}} u)(Z)]g(Y,U) \tag{27}$$

From (27), we have

$$'R(\overline{X},Y,Z,\overline{U}) =' R(X,\overline{Y},\overline{Z},U)$$

if and only if

$$(D_{\overline{Y}} u)(\overline{Z}) = (D_Y u)(Z)$$

Hence the theorem.

Theorem 4.2 *Let V_n be a Kähler manifold equipped with a quarter-symmetric non-metric connection B whose curvature tensor vanishes, then the $1-$form u is a gradient vector field if and only if the manifold is flat.*

Proof In view of (5), (17) becomes

$$R(X,Y,Z) = K(X,Y,Z) + (D_X u)(Z)(\overline{Y}) - (D_Y u)(Z)(\overline{X}) \tag{28}$$

Since $R(X,Y,Z) = 0$, therefore (28) becomes

$$K(X,Y,Z) = (D_Y u)(Z)(\overline{X}) - (D_X u)(Z)(\overline{Y}) \tag{29}$$

If the manifold is flat, then

$$(D_Y u)(Z)(\overline{X}) = (D_X u)(Z)(\overline{Y}) \tag{30}$$

$\Rightarrow$ u is a gradient vector field.

Converse part is obvious from (29) and (30).

Theorem 4.3 *In a Kähler manifold, endowed with a quarter-symmetric non-metric connection B, if the curvature tensor with respect to B vanishes, then we have*

$$K(X,Y,Z) = Ric(\overline{Y}, Z)\overline{X} - Ric(\overline{X}, Z)\overline{Y}.$$

Proof Contracting (29) with respect to X, we get

$$Ric(Y,Z) = -(D_{\overline{Y}} u)(Z)$$

Barring Y in the above relation and then using (1) (a), we get

$$Ric(\overline{Y}, Z) = (D_Y u)(Z) \tag{31}$$

In consequence of (31), (29) gives the required result.

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