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Semi-Symmetric Metric Connections, Einstein Manifolds and Projective Curvature Tensor

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Abstract. Riemananian Manifolds admitting a semi-symmetric metric connection are an important class of manifolds. Such connections give rise to interesting generalizations of Einstein manifolds and projective curvature tensor.We also demonstrate the close relationship between them.

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1. **Introduction**

Friedmann and Schouten [1] first introduced the concept of a semi-symmetric metric connection in 1924. In this and the next section we give a brief recollection of the concepts of Riemannian Manifold theory and Section 3 deals with semi-symmetric metric connections to establish the notation. We follow closely the treatment in [2,3]. Section 4 introduces the idea of a projective curvature tensor with respect to a semi-symmetric metric connection.

Let M be a C^∞ Riemannian manifold of dimension $n \geq 3$ with a positive definite metric tensor g(0,2). In what follows M always stands for such a manifold and X,Y,Z are C^∞ vectorfields , unless otherwise stated. A linear connection on M is a rule ∇ which assigns to each vector field X on M a linear mapping ∇_X satisfying

$$(i)\nabla_{fX+gY} = f\nabla_X + g\nabla_Y \tag{1.1}$$

(1.2)

where f, g are C^∞ functions on M . ∇_X is called covariant differentiation with respect to X. In what follows all the functions are also assumed to be C^∞ unless specified otherwise.

For vector fields X, Y on M the torsion tensor field T of the connection ∇ is defined by

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y] \tag{1.3}$$

The curvature tensor R is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{1.4}$$

A linear connection on (M, g) is called a metric connection if,for any vector field X, it satisfies the condition:

$$\nabla_X g = 0 \tag{1.5}$$

A Riemannian connection ∇ on (M, g) is a metric connection satisfying the following properties:

$$\text{(i) } \nabla_X(Y) - \nabla_Y(X) = [X,Y] \tag{1.6}$$

(1.7)

The curvature tensor R satisfies the following identities:

$$R(X,Y)Z = -R(Y,X)Z \tag{1.8}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \tag{1.9}$$

$$(\nabla_X R)(Y,Z)U + (\nabla_Y R)(Z,X)U + (\nabla_Z R)(X,Y)U = 0 \tag{1.10}$$

The last two equations are called the first and second Bianchi identities respectively.

We also define

$$R(X,Y,Z,U) = g(R(X,Y)Z,U) \tag{1.11}$$

R so defined is called the Riemann-Christoffel curvature tensor of the first kind and is a tensor of the type (0,4).

The Ricci tensor is defined as

$$S(Y,Z) = (C_1^1 R)(Y,Z) = \sum_{i=1}^{n} R(E_i, Y, Z, E_i) \tag{1.12}$$

where $E_i, i = 1.....n$ are orthonormal vector fields on M. Here C denotes the usual contraction of tensors. Ricci tensor is a tensor field of type of (0,2) and is symmetric, that is, $S(X,Y) = S(Y,X)$. The scalar curvature of the Riemannian manifold (M,g) is defined by

$$r = \sum_{i=1}^{n} S(E_i, E_i) \tag{1.13}$$

A Riemannian manifold is said to be flat if

$$R \equiv 0 \tag{1.14}$$

on M

A Riemannian manifold M is said to be an Einstein manifold if

$$S(X,Y) \equiv \mu g(X,Y) \tag{1.15}$$

on M,where μ is a scalar function on M.

Let $X, Y \in T_p M$ at a point $p \in M$. Let γ be the plane spanned by X,Y. Then the sectional curvature w.r.t. the section γ is defined by

$$k(\gamma) = -\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{1.16}$$

Sectional curvature $k(\gamma)$ is uniquely determined by the section γ and is independent of the vectors X, Y in the section. If the sectional curvature $k(\gamma)$ is a constant for all sections γ at each point of M, then M is said to be a space of constant curvature. This follows from [2]

Theorem 1.1. ***(Schur's Theorem)****: If at each point p of a Riemannian manifold M the sectional curvature is independent of the section chosen then M is a space of constant curvature*

If M is a space of constant curvature , then we have

$$R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \tag{1.17}$$

for any vector fields X,Y,Z on M and k is a constant.

2. Conformal Curvature Tensor

Let M be a Riemannian manifold admitting two metric tensors g and $\overline{g}$ such that $\overline{g} = e^{2\rho}g$, where ρ is a function on M. Then $\overline{g}$ is said to be a conformal transformation of g and M is said to be a conformal manifold. The Weyl conformal curvature tensor of a Riemannian manifold M is defined by

(2.1)
$$C(X,Y)Z = R(X,Y)Z) + \lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)L(X) - g(X,Z)L(Y)$$

where

$$\lambda(X,Y) = -\frac{S(X,Y)}{(n-2)} + \frac{rg(X,Y)}{2(n-1)(n-2)} \tag{2.2}$$

and L(X)is defined by

$$\lambda(X,Y) = g(L(X),Y) \tag{2.3}$$

If M is a conformal manifold then

$$\overline{C} = C \tag{2.4}$$

where $\overline{C}$ and C are conformal curvature tensors w.r.t. metrics $\overline{g}$ and g respectively. A Riemannian manifold M ($n \geq 4$) is said to be conformally flat if

$$C \equiv 0 \tag{2.5}$$

on M. Further we state the following theorems here [2]:

Theorem 2.1. *A flat Riemannian manifold is conformally flat*

Theorem 2.2. *A Riemannian manfiold of constant curvature is conformally flat*

Theorem 2.3. *An Einstein manifold conformal to a flat manifold is of constant curvature*

3. Semi-symmetric Metric Connections

Let M be a given manifold with a Riemannian connection ∇. A linear connection $\overline{\nabla}$ is said to be semi-symmetric if the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ satisfies

$$\overline{T} = \pi(Y)X - \pi(X)Y \tag{3.1}$$

where X,Y are vector fields on M and π is a 1-form on M. If $\overline{\nabla}$ further satisfies the condition

$$\overline{\nabla}_X g = 0 \tag{3.2}$$

then $\overline{\nabla}$ is called a semi-symmetric metric connection(s.s.m.c.) [1,3]. Yano[4] obtained a relationship between the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection ∇ which is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P \tag{3.3}$$

where P is a vector field given by

$$g(X,P) = \pi(X) \tag{3.4}$$

The curvature tensor of the s.s.m.c. is defined as

$$\overline{R}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z \tag{3.5}$$

A relation between $\overline{R}$ and R was obtained by Yano[4] given by

(3.6)
$$\overline{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)AX + g(X,Z)AY$$

for any vector fields X,Y and Z where α is a tensor field of type (0,2) defined by

$$\alpha(X,Y) = (\overline{\nabla}_X \pi)(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(P)g(X,Y) \tag{3.7}$$

and A is a tensor field of type (1,1) defined by

$$g(AX,Y) = \alpha(X,Y). \tag{3.8}$$

Note that

$$\overline{R}(X,Y)Z = -\overline{R}(Y,X)Z \tag{3.9}$$

The Ricci tensor $\overline{S}$ and scalar curvature $\overline{r}$ w.r.t. the s.s.m.c are defined analogous to those of the Riemannian connection :

$$\overline{S}(X,Y) = (C_1^1 \overline{R})(Y,Z) = \sum_{i=1}^{n} \overline{R}(E_i, Y, Z, E_i) \tag{3.10}$$

$$\overline{r} = \sum_{i=1}^{n} \overline{S}(E_i, E_i) \tag{3.11}$$

where $E_i, i = 1.....n$ are orthonormal vector fields on M.

Yano [4] proved the following important existence theorem :

Theorem 3.1. *In order that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold is conformally flat.*

Assuming the 1-form π closed i.e. $d\pi = 0$ we proved the following results [5]

Theorem 3.2. *Let (M,g) be a Riemannian manifold admitting a s.s.m.c. Then the Ricci tensor w.r.t. the s.s.m.c is symmetric iff the 1-form π is closed on M*

Theorem 3.3. *Let (M,g) be a Riemannian manifold admitting a s.s.m.c and the 1-form π closed. Then we have:*
$(i)\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0$
$(ii)\overline{R}(X,Y,Z,U) = \overline{R}(Z,U,X,Y)$

Let (M,g) be a Riemannian manifold admitting a s.s.m.c $\overline{\nabla}$ and the 1-form π be closed. We define the sectional curvature w.r.t. $\overline{\nabla}$ for a section spanned by vectors X,Y at a point $p \in M$ by

$$\overline{k} = -\frac{\overline{R}(X,Y,X,Y)}{g(X,Y)g(Y,Y) - g(X,Y)^2} \tag{3.12}$$

We have a definition:

Definition 3.4. A Riemannian manifold is said to be sectionally independent if the sectional curvature $\overline{k}$ w.r.t.the s.s.m.c.is independent of the the section chosen at each point of the manifold

The following result was obtained in [5]

Theorem 3.5. ***Generalized Schur's Theorem*** *Let (M,g) be a Riemannian manifold admitting a s.s.m.c with the 1-form π is closed on M. If the sectional curvature w.r.t. the s.s.m.c. is independent of the section chosen at each point of M then M is not necessarily a space of constant curvature w.r.t. the s.s.m.c.*

Definition 3.6. Let M be a Riemannian manifold admitting a s.s.m.c. M is said to be an Einstein manifold w.r.t. the s.s.m.c. if

$$\overline{S}(X,Y) = \mu g(X,Y) \tag{3.13}$$

for any vector fields X,Y where μ is a scalar function on M.

Clearly $\overline{S}$ is symmetric. Hence from Thm.3.2 the 1-form π is closed.
In [5] we proved the following

Theorem 3.7. *If M is an Einstein manifold w.r.t. a s.s.m.c. then M is conformally flat if and only if M is sectionally independent.*

4. Projective Curvature Tensor with respect to a Semi-symmetric Metric Connection

Let (M,g) be a Riemannian manifold admitting a s.s.m.c. $\overline{\nabla}$. We define the projective curvature tensor $\overline{P}$ w.r.t. the s.s.m.c. $\overline{\nabla}$ by

$$\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-1}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y] \tag{4.1}$$

Hence we have

$\overline{P}(Y,X)Z = \overline{R}(Y,X)Z - \frac{1}{n-1}[\overline{S}(X,Z)Y - \overline{S}(Y,Z)X]$
$= -\overline{P}(X,Y)Z$

Also it is easy to check that

$C_1^1(\overline{P})(Y,Z) = 0$ and if π is closed then $C_3^1(\overline{P})(X,Y) = 0$

Let π be closed. Then we have

$\overline{P}(X,Y)Z + \overline{P}(Y,Z)X + \overline{P}(Z,X)Y$

$= \overline{R}(X,Y)Z - \frac{1}{n-1}[\overline{S}(X,Z)Y - \overline{S}(Y,Z)X] + \overline{R}(Y,Z)X - \frac{1}{n-1}[\overline{S}(Z,X)Y - \overline{S}(Y,X)Z] + \overline{R}(Z,X)Y - \frac{1}{n-1}[\overline{S}(X,Y)Z - \overline{S}(Z,Y)X]$

$= 0$, by Theorems 3.2 and 3.3.

Hence we have proved the

Theorem 4.1. *Let (M,g) be a Riemannian manifold admitting a s.s.m.c.*$\overline{\nabla}$*. Then the projective curvature tensor* $\overline{P}$ *w.r.t. the s.s.m.c.satisfies the following properties:*

(i)$\overline{P}(X,Y)Z = -\overline{P}(Y,X)Z$
(ii) $C_1^1\overline{P})(Y,Z) = 0$*, and if* π *is closed then*
(iii) $C_3^1\overline{P})(X,Y) = 0$
(iv)$\overline{P}(X,Y)Z + \overline{P}(Y,Z)X + \overline{P}(Z,X)Y = 0$

.

We define a Riemannian manifold admitting a s.s.m.c. to be projectively flat if

$$\overline{P}(X,Y)Z = 0, \tag{4.2}$$

for any choice of vector fields X,Y,Z on M.

We also prove the following

Theorem 4.2. *Let (M,g) be a Einstein manifold admitting a s.s.m.c.*$\overline{\nabla}$*. Then M is projectively flat w.r.t. the s.s.m.c if and only if M is sectionally independent*

Proof: Given M is an Einstein manifold w.r.t. the s.s.m.c.. Let M be projectively flat w.r.t. the s.s.m.c.. Then from (4.1), (4.2) and (3.13) we have

$\overline{R}(X,Y)Z = \frac{1}{n-1}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y]$

$= \frac{\mu}{n-1}[g(Y,Z)X - g(X,Z)Y]$

and hence by definition M is sectionally independent.

Conversely, let M be sectionally independent. Then

$$\overline{R}(X,Y)Z = \overline{k}[g(Y,Z)X - g(X,Z)Y] \tag{4.3}$$

where $\overline{k}$ is the sectional curvature w.r.t. the s.s.m.c.. Contracting w.r.t. X and using (1.10) we have

$\overline{S}(Y,Z) = \overline{k}(n-1)g(Y,Z)$

Substituting in (4.3) and noting that $\mu = \overline{k}(n-1)$ we get

$\overline{R}(X,Y)Z = \frac{1}{n-1}[\mu g(Y,Z)X - \mu g(X,Z)Y]$

$= \frac{1}{n-1}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y]$

From (4.1) we obtain

$\overline{P}(X,Y)Z = 0$

From Theorems (3.7) and (4.2) we immediately have the following

Theorem 4.3. *If M is an Einstein manifold w.r.t. a s.s.m.c., then M is projectively flat w.r.t. to the s.s.m.c. if and only if M is a conformally flat manifold.*

For a different approach to the projective curvature tensor w.r.t. a s.s.m.c. see ref [6,7].

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