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Strongly αg*-Closed Sets in Bitopological Spaces

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Abstract

We introduce strongly αg*-closed set in bitopological spaces and some properties of these sets are investigated. Three bitopological spaces (i, j)-$_{s*}T_c$, (i, j)-$_{s*}T_g$ and (i, j)-$_{s*}T_{g*}$ spaces are introduced as applications. A characterization for the topological space (i, j)-$_{s*}T_c$ is also introduced.

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1. INTRODUCTION

A triplet (X, τ_1, τ_2), where X is a non-empty set and τ_1 and τ_2 are topologies on X, is called a bitopological space and Kelly [2] initiated the study of such spaces. In 1985, Fukutake [1] introduced the concepts of g- closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces. Sheik John and Sundaram [7] introduced g*- closed sets and g*-continuity in bitopological spaces in 2002.

Recently the authors introduced and studied sαg*-closed sets [4] in topological spaces. In this paper we introduce the concepts of sαg*-closed sets, $_{s*}T_c$ space, $_{s*}T_g$ spaces and $_{s*}T_{g*}$ space for bitopological spaces and investigate some of their properties.

2. PRELIMINARIES

If A is a subset of a topological space (X, τ), then the closure of A is denoted by τ-cl(A) or cl(A), the interior of A is denoted by τ-int(A) or int(A) and the complement of A in X is denoted by A^c.

***Definition 2.1*:** A subset A of a topological space (X, τ) is called
(i) an α–open set [6] if $A \subseteq$ int(cl(int(A))) and an α–closed set [5] if cl(int(cl(A))) $\subseteq$A.
(ii) a generalized closed set [3] (briefly g-closed set) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.
(iii) a generalized open set [3] (briefly g-open set) if A^c is g-closed in X.
(iv) a g*-closed set [8] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in (X, τ).
(v) a sαg*-closed set [4] if αCl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g*-open in X.
(vi) a sαg*-open set [4] if A^c is a s αg*-closed set.

Definition 2.2: The class of all α–open subsets of a space (X, τ) is denoted by τ^α. The intersection of all α-closed sets containing a subset A of (X, τ) is called the α-closure of A and is denoted by αcl (A) or τ -αcl(A).

Throughout this paper X and Y always represent non-empty bitopological spaces (X, τ_1, τ_2) and (Y, σ_1 , σ_2) on which no separation axioms are assumed unless explicitly mentioned and the integers i, j, k $\in$ {1,2}. For a subset A of X, τ_i -cl(A) (resp.τ_i -int(A), τ_i -αcl (A)) denote the closure (resp. interior, α-closure) of A with respect to the topology τ_i . We denote the family of all g-open and g*-open subsets of X with respect to the topology τ_i by GO(X, τ_i) and G*O (X, τ_i) respectively and the family of all τ_j -closed sets is denoted by the symbol F_j. By (i, j) we mean the pair of topologies (τ_i , τ_j).

***Definition 2.3*:** A subset A of a topological space (X, τ_1 ,τ_2) is called
(i) (i, j)-g-closed [1] if τ_j -cl(A) $\subseteq$ U whenever A $\subseteq$ U and U $\in$ τ_i.
(ii) (i, j)-g*-closed set [7] if τ_j -cl(A) $\subseteq$ U whenever A $\subseteq$ U and U$\in$GO(X, τ_i)

The family of all (i, j)-g-closed (resp.(i, j)-g*-closed) subsets of a bitopological space (X, τ_1, τ_2) is denoted by D(i, j)(resp. D*(i, j)).

Definition 2.4: (i) A bitopological space (X, τ_1, τ_2) is said to be (i, j)-$T_{½}$ [1] if every (i, j) g-closed set is τ_j -closed.

(ii) A bitopological space (X, τ_1, τ_2) is said to be s pairwise $T_{½}$ [1] if it is (1, 2)-$T_{½}$ and (2,1)-$T_{½}$.

(iii) A bitopological space (X, τ_1, τ_2) is said to be an (i, j)-$T^*_{½}$ space [7] if every (i, j)-g*-closed set is τ_j -closed.

(iv) A bitopological space (X, τ_1, τ_2) is said to be s pairwise $T^*_{½}$ space [7] if it is both (1, 2)-$T^*_{½}$ and (2,1)-$T^*_{½}$.

(v) A bitopological space (X, τ_1, τ_2) is said to be an (i, j)-$^*T_{½}$ space [7] if every (i, j)-g-closed set is (i, j)-g*-closed .

3. (i, j)-STRONGLY αg*-CLOSED SETS

In this section we introduce the concept of (i, j)-strongly αg*-closed sets in bitopological spaces and discuss some of their properties.

Definition 3.1: A subset A of a bitopological space (X, τ_1, τ_2) is said to be an (i, j)-strongly αg*-closed(briefly (i, j)-sαg*-closed) set if τ_j-αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U $\in$ G*O(X, τ_i).

We denote the family of all (i, j)-sαg*-closed sets in (X, τ_1, τ_2) by S*(i, j).

***Remark 3.2*:** By setting $\tau_1 = \tau_2$ in Definition 3.1, a (i, j)-sαg*-closed set is a sαg*-closed set.

Proposition 3.3: If A is τ_j-closed subset of (X, τ_1, τ_2) then A is (i, j)-sαg*-closed set.

The converse of the above proposition is not true as seen from the following example.

***Example 3.4*:** Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then the subset {a, b} is a (1, 2) sαg*-closed set but not τ_2 -closed in (X, τ_1, τ_2).

***Proposition 3.5*:** If A is τ_j -g*-closed subset of (X, τ_1, τ_2) then A is (i, j)- sαg*-closed set.

The converse of the above proposition is not true always. The subset {a, b} in Example 3.4 is (1, 2)-sαg*-closed set but not τ_2 -g* closed set in (X, τ_1, τ_2)

Proposition 3.6: If A is both τ_j -g*-open and (i, j)-sαg*-closed, then A is τ_j α-closed.

Proposition 3.7: If A is (i, j)-g*-closed subset of (X, τ_1, τ_2) then A is (i, j)- sαg*-closed set.

The converse of the above proposition is not true always. The subset {a, b} in the above Example 3.4 is (1, 2)-sαg*-closed set but not (1, 2)-g*-closed set in (X, τ_1, τ_2)

Proposition 3.8: If A, B$\in$ S*(i, j) then A $\cup$ B $\in$ S* (i, j).

Remark 3.9: The intersection of two (i, j)-sαg*-closed sets need not be an (i, j)-sαg*-closed set as seen from the following example.

***Example 3.10*:** If X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b, c}, X} then {a, b} and {a, c} are (2,1)- sαg*-closed sets but {a, b} $\cap$ {a, c} = {a} is not a (2,1)-sαg*-closed set.

Remark 3.11: S* (1, 2) is generally not equal to S* (2, 1).
In Example 3.4, S* (1, 2) = P(X) and S* (2, 1) = P(X) – {a}. Therefore, S* (2, 1) $\neq$S* (1, 2)

Proposition 3.12: If $\tau_1 \subseteq \tau_2$ in (X, τ_1, τ_2) then S*(2,1) $\subseteq$ S*(1,2).
The converse of the above proposition is not true as seen from the following example.

***Example 3.13*:** Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a, c}, X}. Then S*(2, 1) $\subseteq$ S* (1, 2) but τ_1 is not contained in τ_2.

Proposition 3.14: If A is (i, j)-sαg*-closed then τ_j-αcl(A)–A contains no non-empty τ_i -g* closed set.

Proof: Let A be an (i, j)-sαg*-closed set and F be a non-empty τ_i-g*-closed subset such that F $\subseteq$ τ_j-αcl(A) – A = τ_j-αcl(A) $\cap$ A^c . Therefore, F $\subseteq$ τ_j-α cl(A) and F $\subseteq$ A^c. Since F^c is a τ_i -g*-open and A is (i, j)-sαg*-closed set in (X, τ), we have τ_j-αcl(A) $\subseteq$ F^c. That is, F $\subseteq$ (τ_j- αcl(A))^c. Hence F $\subseteq$ (τ_j- αcl(A) $\cap$ (τ_j-αcl(A))^c = ϕ. Therefore, αcl (A) – A contains no non-empty τ_j -g*-closed set.
The converse of the above proposition is not true as seen from the following example.

Example 3.15: Let X = {a, b, c, d}, τ_1 = {ϕ, {a}, {a, d}, X} and τ_2 = {ϕ, X, {a, b}, {c, d}}. If A= {a} then τ_2-αcl (A) – A = {b} does not contain any non- empty τ_1 -g*-closed set. But A is not a (1, 2)-sαg*-closed set.

Corollary 3.16: If A is (i, j)-sαg*-closed set in (X, τ_1, τ_2), then A is τ_j -α-closed if and only if τ_j-αcl(A) – A is τ_i -g*-closed.

Proof: Necessity: If A is τ_j-α-closed, then τ_j-αcl(A) = A. i.e., τ_j-αcl(A) – A = ϕ and hence τ_j-αcl(A) – A is τ_j -g* closed.

Sufficiency: If τ_j -αcl(A) – A is τ_j – g*-closed then by Proposition 3.14, τ_j- αcl(A) – A = ϕ, since A is (i, j)-sαg*-closed set. Therefore, A is τ_j - α- closed.

Proposition 3.17: If A is an (i, j)-sαg*-closed set of (X,τ_i,τ_j) such that A $\subseteq$ B $\subseteq$ τ_j-αcl(A), then B is also an (i, j)-sαg*-closed set of (X, τ_i, τ_j).

Proof: Let U be a τ_j-g*-open set of (X, τ_1, τ_2) such that B $\subseteq$ U. Then, A $\subseteq$ U. Since A is a (i, j)-sαg*-closed set, τ_j -αcl(A) $\subseteq$ U. Also τ_j-αcl(B) $\subseteq$ τ_j-αcl(τ_j-αcl(A)) = τ_j-αcl(A) $\subseteq$ U which implies, τ_j-αcl(B) $\subseteq$ U. Hence B is also an (i, j)-sαg*-closed set of (X, τ_i, τ_j).

Proposition 3.18: For each element x of (X, τ_1, τ_2), {x} is τ_i-g*-closed or $\{x\}^c$ is (i, j)-sαg*-closed.

Proposition 3.19: In a bitopological space (X, τ_1, τ_2) if G*O(X, τ_i) $\subseteq$ F_j then every subset of X is an (i, j)-sαg*-closed set of (X, τ_i, τ_j).

Proof: Let G*O(X, τ_i) $\subseteq$ F_j . Let A be a subset of X such that A $\subseteq$ U where U G*O(X, τ_i). Then τ_j -αcl(A) $\subseteq$ τ_j -cl(A) $\subseteq$ τ_j -cl(U) $\subseteq$ U and hence A is an (i, j)-sαg*-closed set of (X, τ_i, τ_j).

4. (i, j)-$_{s*}T_c$, (i, j)- $_{s*}T_g$ AND (i, j)-$_{s*}T_{g*}$ SPACES

In this section, we introduce (i, j)- $_{s*}T_c$, (i, j)- $_{s*}T_{g*}$ and (i, j)- $_{s*}T_g$ bitopological spaces and in Theorem 4.19 we prove that the (i, j)- $_{s*}T_{g*}$ space is the dual of the class of (i, j)-$T^*_{1/2}$ space to the class of (i, j)-$_{s*}T_c$ space.

Definition 4.1: A bitopological space (X, τ_1, τ_2) is said to be an (i, j)-$_{s*}T_c$ space if every (i, j)-sαg*-closed set is τ_j -closed.

Proposition 4.2: If (X, τ_1, τ_2) is (i, j)-$_{s*}T_c$ space, then it is an (i, j)-$T^*_{½}$ space but not conversely.

Example 4.3: Let X = {a, b, c}, τ_1 = { ϕ, {b}, X} and τ_2 = {ϕ, {b}, {a, c}, X}. Then (X, τ_1, τ_2) is a (1, 2)-$T^*_{½}$ space but not a (1, 2)-$_{s*}T_c$ space.

Theorem 4.4: A bitopological space (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space if and only if {x} is τ_j -open or τ_i -g*-closed for each x $\in$ X.

Proof: Suppose that {x} is not τ_i -g*-closed. Then by Proposition 3.18, $\{x\}^c$ is (i, j)-s αg*-closed set. Since (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space, $\{x\}^c$ is τ_j -closed. Therefore, {x} is τ_j -open.

Conversely, let F be an (i, j)-sαg*-closed set. By assumption, {x} is τ_j -open or τ_i -g*-closed for x $\in$ τ_j-αcl(F) $\subseteq$ τ_j-cl(F).

Case (i) : Suppose $\{x\}$ is τ_j -open. Since $\{x\} \cap F \neq \phi$, we have $x \in F$.
Case (ii) : Suppose $\{x\}$ is τ_i -g*-closed. If $x \notin F$, then $\{x\} \subseteq \tau_j$-αcl(F) – F, which is a contradiction to Proposition 3.14. Therefore, $x \in F$.

Thus in both cases, we conclude that F is τ_j -closed. Hence (X, τ_1, τ_2) is an (i, j)-${}_{s*}T_c$ space.

Remark 4.5: (X, τ_1) space is not generally a ${}_{s*}T_c$ space, even if (X, τ_1, τ_2) is (1, 2)-${}_{s*}T_c$ space as shown in the following Example 4.6. Also (X, τ_1, τ_2) is not generally a (1, 2)-${}_{s*}T_c$ space even if both (X, τ_1) and (X, τ_2) are ${}_{s*}T_c$ space. This is shown in Example 4.7.

Example 4.6: Let X = {a, b, c}, τ_1 = {ϕ, {b}, {b, c}, X} and τ_2 = {ϕ, {b}, {c}, {b, c}, X}. Then (X, τ_1) is not a ${}_{s*}T_c$ space but (X, τ_1, τ_2) is a (1, 2)-${}_{s*}T_c$ space.

Example 4.7: Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, c}, X} and τ_2 = {ϕ, {a, b}, X}. Then both (X, τ_1) and (X, τ_2) are ${}_{s*}T_c$ spaces but (X, τ_1, τ_2) is not a (1, 2)- ${}_{s*}T_c$ space.

Definition 4.8: A bitopological space (X, τ_1, τ_2) is said to be spairwise ${}_{s*}T_c$ space if it is both (1, 2)-${}_{s*}T_c$ and (2,1)-${}_{s*}T_c$ space.

Proposition 4.9: If (X, τ_1, τ_2) is spairwise ${}_{s*}T_c$ space then it is spairwise $T^*_{½}$ space but not conversely.

Example 4.10: Let (X, τ_1, τ_2) be as in Example 4.3. Then, (X, τ_1, τ_2) is a spairwise $T^*_{½}$ space. But (X, τ_1, τ_2) is not a spairwise ${}_{s*}T_c$ space since it is not a (2,1)-${}_{s*}T_c$ space .

We introduce the following definition.

Definition 4.11: A bitopological space (X, τ_1, τ_2) is said to be an (i, j)-${}_{s*}T_g$ space if every (i, j)- sαg*-closed is (i, j)-g-closed set.

Proposition 4.12: Every (i, j)-${}_{s*}T_c$ space is an (i, j)-${}_{s*}T_g$ space but not conversely.

Example 4.13: Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, c}, X} and τ_2 = {ϕ, {a, b}, X}. Then (X, τ_1, τ_2) is a (2, 1)- ${}_{s*}T_g$ space but not a (2, 1)- ${}_{s*}T_c$ space.

Proposition 4.14: Every (i, j)- ${}_{s*}T_{g*}$ space is an (i, j)- ${}_{s*}T_g$ space but not conversely.

Example 4.15: Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a, c}, X}. Then (X, τ_1, τ_2) is a (1, 2)- ${}_{s*}T_g$ space but not a (1, 2)- ${}_{s*}T_{g*}$ space.

Theorem 4.16: A bitopological space (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space if and only if it is both (i, j)-$_{s*}T_g$ space and (i, j)-$T^*_{½}$ space.

Proof: Suppose that (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space. Then by Proposition 4.2 and 4.12, (X, τ_1, τ_2) is an (i, j)-$T^*_{½}$ space and (i, j)- $_{s*}T_g$ space.

Conversely, suppose (X, τ_1, τ_2) is both (i, j)-$_{s*}T_g$ space and (i, j)-$T^*_{½}$ space. Let A be an (i, j)- sαg*-closed set of (X, τ_1, τ_2). Since (X, τ_1, τ_2) is an (i, j)-$_{s*}T_g$ space, A is an (i, j)-g-closed set. Since (X, τ_1, τ_2) is an (i, j)- $T^*_{½}$ space, A is τ_j-closed set of (X, τ_1, τ_2). Therefore, (X, τ_1, τ_2) is an (i, j)- $_{s*}T_c$ space.

We introduce the following definition.

Definition 4.17: A bitopological space (X, τ_1, τ_2) is said to be an (i, j)-$_{s*}T_{g*}$ space if every (i, j)-sαg*-closed set is an (i, j)-g*-closed set.

***Proposition 4.18*:** Every (i, j)-$_{s*}T_c$ space is an (i, j)-$_{s*}T_{g*}$space but not conversely.

Let (X, τ_1, τ_2) be as in Example 4.3. Then, (X, τ_1, τ_2) is an (i, j)-$_{s*}T_{g*}$ space but not a (1, 2)-$_{s*}T_c$ space.

***Theorem 4.19*:** A bitopological space (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space if and only if it is both (i, j)-$_{s*}T_{g*}$ space and (i, j)-$T^*_{½}$ space.

Proof: Suppose that (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space. Then by Proposition 4.2 and 4.18, (X, τ_1, τ_2) is an (i, j)-$T^*_{½}$ space and (i, j)-$_{s*}T_{g*}$ space.

Conversely, suppose that (X, τ_1, τ_2) is both (i, j)-$T^*_{½}$ space and (i, j) -$_{s*}T_{g*}$space. Let A be an (i, j)-sαg*-closed set of (X, τ_1, τ_2). Since (X, τ_1, τ_2) is an (i, j)-$_{s*}T_{g*}$ space, A is an (i, j)-g* closed set. Since (X, τ_1, τ_2) is an (i, j)-$T^*_{½}$ space, A is τ_j-closed set of (X, τ_1, τ_2). Therefore, (X, τ_1, τ_2) is an (i, j)-$_{s*}T_c$ space.

***Remark 4.20*:** (i, j)-$T^*_{½}$ and (i, j)-$_{s*}T_{g*}$ spaces are independent as seen from the following two examples.

***Example 4.21*:** Let X, τ_1 and τ_2 be as in Example 4.3. Then (X, τ_1, τ_2) is a (i, j)-$T^*_{½}$ space but it is not an (i, j)-$_{s*}T_{g*}$ space.

***Example 4.22*:** Let X, τ_1 and τ_2 be as in Example 4.13. Then (X, τ_1, τ_2) is an (i, j)-$_{s*}T_{g*}$ space but it is not an (i, j)-$T^*_{½}$ space.

***Remark 4.23*:** In an (i, j)-$_{s*}T_{g*}$ space (X, τ_1, τ_2), the converse of the Theorem 3.19 is true.

Remark 4.24: (i, j)-$T^*_{½}$ and (i, j)-$_{s*}T_g$ spaces are independent as seen from the following two examples.

Example 4.25: Let X = {a, b, c,d}, τ_1 = {ϕ, {a}, {a, d}, X} and τ_2 = {ϕ, {a, b}, {c, d}, X}. Then (X, τ_1, τ_2) is a (2, 1)- $T^*_{½}$ space but not a (2, 1)-$_{s*}T_g$ space. Also, (X, τ_1, τ_2) is a (1, 2)- $_{s*}T_g$ space but not a (1, 2)- $T^*_{½}$ space.

Remark 4.26: **(**i, j)-$T_{½}$ and (i, j)-$_{s*}T_g$ spaces are independent as seen from the following two examples.

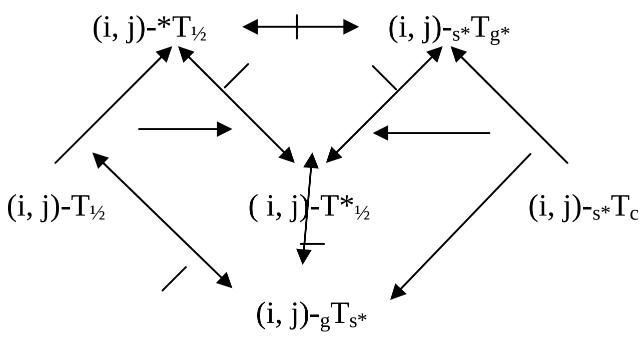
Example 4.27**:** Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, c}, X} and τ_2 = {ϕ, {a, b}, X}. Then (X, τ_1, τ_2) is a (2, 1)-$_{s*}T_g$ space but not a (2, 1)-$T_{½}$ space.

Example 4.28**:** Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}. Then (X, τ_1, τ_2) is a (2, 1)- $T_{½}$ space but not a (2, 1)- $_{s*}T_g$ space.

Remark 4.29: **(**i, j)-$T^*_{½}$ and (i, j)-$_{s*}T_{g*}$ spaces are independent as seen from the following two examples.

Example 4.30**:** Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then (X, τ_1, τ_2) is a (1, 2)-$T_{½}$ space but not a (1, 2)-$_{s*}T_{g*}$ space. Also, the bitopological space (X, τ_1, τ_2) in Example 4.27 is a (1, 2)- $_{s*}T_{g*}$ space but not a (1, 2)- $T_{½}$ space

Remark 4.31**:** From the above discussions and known results we have the following diagram:



(Where A $\rightarrow$ B (resp. A $\not\leftrightarrow$ B) represents A implies B but not conversely (resp. A and B are independent))

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REFERENCES

[1] T. Fukutake, On generalized closed sets in bitopological spaces, Bull. Fukuoka Univ. Ed. Part III, 35 (1985), 19-28.

[2] J.C. Kelley, Bitopological spaces, Proc. London Math. Sci., 13 (1963), 71 – 89.

[3] N. Levine, Generalized closed sets in topology, Rend. Cire. Math. Palermo, 19 (1970), 89 – 96.

[4] S. Maragatharalli and M. Sheik John, On s αg* - closed sets in topological spaces, ACTA CIENCIA INDICA, Vol XXXI 2005 No.3, (2005), 805 – 814.

[5] S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α-continuous and α-open mappings, Acta Math. Hung., 41(3-4)(1983), 213-218.

[6] O.Njastad, on some classes of nearly open sets, Pacfic J.Math., 15 (1965), 961-970.

[7] M. Sheik John and P. Sundaram, g* - closed sets in bitopological spaces, Indian J. Pure and appl. Math. 35(1), (2004), 71 – 80.

[8] M.K.R.S. Veera Kumar, Between closed sets and g-closed sets, Mem. Fac. Sci. Kochi Univ. (Math.), 21 (2000), 1- 19.

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