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Energy, Laplacian Energy and Zagreb Index of Line Graph, Middle Graph and Total Graph

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Abstract

The energy of a graph G is defined as the sum of the singular values of its adjacency matrix and the Laplacian energy of G is defined as the sum of the distance between Laplacian eigenvalues and average degree of G. We report upper bounds for the energy and Laplacian energy of line graph, middle graph and total graph. The bounds for the Laplacian energy are given using the first Zagreb index of the graph.

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1 Introduction

Let G be a simple graph. Let $\mathbf{A}(G)$ be the adjacency matrix of G. The eigenvalues of G are just the eigenvalues of the matrix $\mathbf{A}(G)$ [1]. The energy $E(G)$ of G is defined as the sum of the absolute values of the eigenvalues of G. This graph invariant was proposed by Gutman [2], found applications in the molecular orbital theory of conjugated π-electron systems [2–4].

Let $\mathbf{D}(G)$ be the degree diagonal matrix of the graph G. Then $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ is the Laplacian matrix of G. Denote by $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$ the Laplacian eigenvalues (i.e., eigenvalues of $\mathbf{L}(G)$) of G, arranged in a non-increasing order, where n is the number of vertices of G, see [8]. The Laplacian energy $LE(G)$ [7, 15] of G is defined as the sum of the distance between Laplacian eigenvalues of G and the average degree $d(G)$ of G, for which more results may be found in [15, 16, 12].

The first Zagreb index [5] of a graph G is defined as $Zg(G) = \sum_{u \in V(G)} d_u^2$, where d_u is the degree of the vertex u in G.

Let $\mathcal{L}_G$ be the line graph of the graph G. The middle graph $\mathcal{M}_G$ is the graph obtain from G by inserting a new vertex (of degree 2) on each edge of G and then joining by edges those pairs of the new added vertices which lie on adjacent edges of G. The total graph $\mathcal{T}_G$ of a graph G is the graph obtain from $\mathcal{M}_G$ by joining pairs of vertices which are adjacent in G by edges.

For a graph G with n vertices and $m \geq 1$ edges, we report a upper bounds for $E(\mathcal{L}_G)$, $E(\mathcal{M}_G)$ and $E(\mathcal{T}_G)$ using m, and upper bounds for $LE(\mathcal{L}_G)$, $LE(\mathcal{M}_G)$ and $LE(\mathcal{T}_G)$ using $Zg(G)$.

2 Preliminaries

The singular eigenvalues of a real matrix $\mathbf{X}$ are the square roots of the eigenvalues of the matrix $\mathbf{X}\mathbf{X}^t$, where $\mathbf{X}^t$ denotes the transpose of the matrix $\mathbf{X}$. For an $n \times n$ matrix $\mathbf{X}$, its singular values are denoted in a non-increasing order by $s_1(\mathbf{X}), s_2(\mathbf{X}), \ldots, s_n(\mathbf{X})$. Then for the graph G with n vertices, we have [11] $E(G) = \sum_{i=1}^n s_i(\mathbf{A}(G))$ and $LE(G) = \sum_{i=1}^n s_i(\mathbf{L}(G) - d(G)\mathbf{I}_n)$.

Lemma 2.1 *[13] Let $\mathbf{X}$ and $\mathbf{Y}$ be $n \times n$ positive semi-definite real matrices. Then $\sum_{i=1}^n s_i(\mathbf{X} - \mathbf{Y}) \leq \sum_{i=1}^n s_i(\mathbf{X} \oplus \mathbf{Y})$, where $\mathbf{X} \oplus \mathbf{Y} = \begin{pmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{Y} \end{pmatrix}$.*

Lemma 2.2 *[9] Let $\mathbf{X}$ and $\mathbf{Y}$ be $n \times n$ real matrices. Then*

$$\sum_{i=1}^n s_i(\mathbf{X} + \mathbf{Y}) \leq \sum_{i=1}^n s_i(\mathbf{X}) + \sum_{i=1}^n s_i(\mathbf{Y}).$$

For the graph G with n vertices, denote by $\mu_1^+(G), \mu_2^+(G), \ldots, \mu_n^+(G)$ the signless Laplacian eigenvalues, i.e., eigenvalues of the signless Laplacian matrix $\mathbf{L}^+(G) = \mathbf{D}(G) + \mathbf{A}(G)$, of G, arranged in a non-increasing order, see [6].

Let $\mathbf{B} = \mathbf{B}(G)$ be the (vertex–edge) incidence matrix of the graph G. Then $\mathbf{B}\mathbf{B}^t = \mathbf{L}^+(G)$ and $\mathbf{B}^t\mathbf{B} = \mathbf{A}(\mathcal{L}_G) + 2I_m$ [8, 6].

3 Line Graph

Let G be a graph with n vertices and $m \geq 1$ edges. $\mathcal{L}_G$ have m vertices and $\frac{Zg(G)}{2} - m$ edges. Then $d(\mathcal{L}_G) = \frac{Zg(G)}{m} - 2$.

Theorem 3.1 *Let G be a graph with n vertices and $m \geq 1$ edges. Then*
(i) $E(\mathcal{L}_G) \leq 4m - 2$; (ii) $LE(\mathcal{L}_G) \leq (2Zg(G) - 4m)\left(1 - \frac{1}{m}\right)$.

Proof. (ii) follows from a result in [12]: $LE(G) \leq 4m\left(1-\frac{1}{n}\right)$. We need only to prove (i). Since $m \geq 1$, we have $\mu_1^+(G) \geq \Delta + 1 \geq 2$, where Δ is the maximum degree of G. From $\mathbf{A}(\mathcal{L}_G) = \mathbf{B}^t\mathbf{B} - 2\mathbf{I}_m$ and by Lemma 2.1,

$$\begin{aligned} E(\mathcal{L}_G) &= \sum_{i=1}^{m} s_i(\mathbf{A}(\mathcal{L}_G)) = \sum_{i=1}^{m} s_i(\mathbf{B}^t\mathbf{B} - 2\mathbf{I}_m) \\ &\leq \sum_{i=1}^{m} s_i(\mathbf{B}^t\mathbf{B} \oplus 2\mathbf{I}_m) = \mu_1^+(G) + \sum_{i=2}^{m} \max\{s_i(\mathbf{B}^t\mathbf{B}), 2\} = 4m-2, \end{aligned}$$

as desired. ■

4 Middle Graph

For a graph G with n vertices and m edges, $\mathcal{M}_G$ have $m+n$ vertices and $\frac{Zg(G)}{2} + m$ edges. Then $d(\mathcal{M}_G(G)) = \frac{Zg(G)+2m}{m+n}$.

Theorem 4.1 *Let G be graph with n vertices and $m \geq 1$ edges. Then*

(i) $E(\mathcal{M}_G) \leq 2\sqrt{2mn} + 4m - 2$;

(ii) $LE(\mathcal{M}_G) \leq 2\sqrt{2mn} + 2Zg(G)$.

Proof. (i) Since $\mathbf{B}\mathbf{B}^t = \mathbf{L}^+(G)$, we have (see also [14])

$$\sum_{i=1}^{m+n} s_i\left(\begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^t & 0 \end{pmatrix}\right) = 2\sum_{i=1}^{n}\sqrt{\mu_i^+(G)}\,.$$

By the Cauchy–Schwarz inequality, we have

$$\sum_{i=1}^{n}\sqrt{\mu_i^+(G)} \leq \sqrt{n\sum_{i=1}^{n}\mu_i^+(G)} = \sqrt{2mn}\,.$$

Note that

$$\mathbf{A}(\mathcal{M}_G(G)) = \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^t & \mathbf{A}(\mathcal{L}_G) \end{pmatrix}.$$

By Lemma 2.2 and Theorem 3.1 (i),

$$\begin{aligned} E(\mathcal{M}_G) &\leq \sum_{i=1}^{m+n} s_i\left(\begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^t & 0 \end{pmatrix}\right) + \sum_{i=1}^{m+n} s_i\left(\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{L}(\mathcal{L}_G) \end{pmatrix}\right) \\ &= 2\sum_{i=1}^{n}\sqrt{\mu_i^+(G)} + E(\mathcal{L}_G) \leq 2\sqrt{2mn} + 4m - 2\,. \end{aligned}$$

(ii) Note that

$$\begin{aligned} & \mathbf{L}(\mathcal{M}_G) - d(\mathcal{M}_G)\mathbf{I}_{m+n} \\ = & \begin{pmatrix} \mathbf{D}(G) - d(\mathcal{M}_G)\mathbf{I}_n & -\mathbf{B} \\ -\mathbf{B}^t & \mathbf{L}(\mathcal{L}_G) - (d(\mathcal{M}_G)-2)\mathbf{I}_m \end{pmatrix}, \end{aligned}$$

$d(\mathcal{M}_G) - 2 \geq 0$ and $\sqrt{\mu_i^+(G)} \leq \sqrt{2mn}$. By Lemma 2.2,

$$\begin{aligned} LE(\mathcal{M}_G) &\leq \sum_{i=1}^{m+n} s_i \left(\begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^t & 0 \end{pmatrix} \right) + \sum_{i=1}^{n} s_i \left(\mathbf{D}(G) - d(\mathcal{M}_G)\mathbf{I}_n \right) \\ &\quad + \sum_{i=1}^{m} s_i \left(\mathbf{L}(\mathcal{L}_G) - (d(\mathcal{M}_G)-2)\mathbf{I}_m \right) \\ &= 2\sum_{i=1}^{n} \sqrt{\mu_i^+(G)} + \sum_{u \in V(G)} |d_u - d(\mathcal{M}_G)| + \sum_{i=1}^{m} |\mu_i(\mathcal{L}_G) - d(\mathcal{M}_G) + 2| \\ &\leq 2\sum_{i=1}^{n} \sqrt{\mu_i^+(G)} + \sum_{u \in V(G)} d_u + nd(\mathcal{M}_G) + \sum_{i=1}^{m} \mu_i(\mathcal{L}_G) + m(d(\mathcal{M}_G) - 2) \\ &\leq 2\sqrt{2mn} + 2Zg(G), \end{aligned}$$

as desired. ■

5 Total Graph

For a graph G with n vertices and m edges, $\mathcal{T}_G$ have $m+n$ vertices and $\frac{Zg(G)}{2} + 2m$ edges. Then $d(\mathcal{T}_G) = \frac{Zg(G)+4m}{m+n}$.

Theorem 5.1 *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

(i) $E(\mathcal{T}_G) \leq 3\sqrt{2mn} + 4m - 2$;

(ii) $LE(\mathcal{T}_G) \leq 2\sqrt{2mn} + +2Zg(G) + 4m$.

Proof. (i) Note that $\mathbf{A}(\mathcal{T}_G) = \begin{pmatrix} \mathbf{A}(G) & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A}(\mathcal{L}_G) \end{pmatrix}$, $\sum_{i=1}^{n} \sqrt{\mu_i^+(G)} \leq \sqrt{2mn}$ and $E(G) \leq \sqrt{2mn}$. By Lemma 2.2,

$$\begin{aligned} E(G) &\leq \sum_{i=1}^{n} s_i \left(\begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^t & 0 \end{pmatrix} \right) + E(G) + E(\mathcal{L}_G) \\ &= 2\sum_{i=1}^{n} \sqrt{\mu_i^+(G)} + E(G) + E(\mathcal{L}_G) \end{aligned}$$

$$\leq \ 2\sum_{i=1}^{n}\sqrt{\mu_i^+(G)} + E(G) + 4m - 2 \leq 3\sqrt{2mn} + 4m - 2\,.$$

(ii) Since G is connected, we have $Zg(G) + 4m \geq 2(m+n)$, and then $d(\mathcal{T}_G) - 2 \geq 0$. By the Lemma 2.2, we have

$$\begin{aligned}\sum_{i=1}^{n} s_i\left(2\mathbf{D}(G) - \mathbf{A}(G) - d(\mathcal{T}_G)\mathbf{I}_n\right) &\leq \sum_{i=1}^{n}\mu_i(G) + \sum_{i=1}^{n} s_i\left(D(G)\right) + nd(\mathcal{T}_G)\\ &= 4m + nd(\mathcal{T}_G)\,.\end{aligned}$$

Note that

$$\mathbf{L}(\mathcal{T}_G) - d(\mathcal{T}_G)\mathbf{I}_{m+n} = \begin{pmatrix} 2\mathbf{D}(G) - \mathbf{A}(G) - d(\mathcal{T}_G)\mathbf{I}_n & -\mathbf{B}(G) \\ -\mathbf{B}^T(G) & \mathbf{L}(\mathcal{L}_G) - (d(\mathcal{T}_G) - 2)\mathbf{I}_m \end{pmatrix}.$$

By Lemma 2.2,

$$\begin{aligned} LE(\mathcal{T}_G) &\leq \sum_{i=1}^{m+n} s_i\left(\begin{pmatrix} 0 & -\mathbf{B}(G) \\ -\mathbf{B}^T(G) & 0 \end{pmatrix}\right) + \sum_{i=1}^{n} s_i\left(2\mathbf{D}(G) - \mathbf{A}(G) - d(\mathcal{T}_G)\mathbf{I}_n\right)\\ &\quad + \sum_{i=1}^{m} s_i\left(\mathbf{L}(\mathcal{L}_G) - (d(\mathcal{T}_G) - 2)\mathbf{I}_m\right)\\ &\leq 2\sum_{i-1}^{n}\sqrt{\mu_i^+(G)} + 4m + nd(\mathcal{T}_G) + \sum_{i=1}^{m-1} s_i\left(\mathbf{L}(\mathcal{L}_G) - (d(\mathcal{T}_G) - 2)\mathbf{I}_m\right)\\ &\quad + d(\mathcal{T}_G) - 2\\ &\leq 2\sum_{i-1}^{n}\sqrt{\mu_i^+(G)} + 4m + nd(\mathcal{T}_G) + \sum_{i=1}^{m-1}\max\left\{\mu_i(\mathcal{L}_G), d(\mathcal{T}_G) - 2\right\}\\ &\quad + d(\mathcal{T}_G) - 2\\ &\leq 2\sum_{i-1}^{n}\sqrt{\mu_i^+(G)} + 4m + nd(\mathcal{T}_G) + \sum_{i=1}^{m-1}\mu_i(\mathcal{L}_G) + md(\mathcal{T}_G) - 2m\\ &\leq 2\sqrt{2mn} + 2Zg(G) + 4m, \end{aligned}$$

as desired. ■

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