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Application of the Modified Homotopy Perturbation Method to the Two Dimensional sine-Gordon Equation

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Abstract

In this paper, we apply the modified homotopy perturbation method based on the Taylor series for solving the initial value problems of the two dimensional sine-Gordon equation. Due to the proposed modification, the approximate solution obtained is in accordance with the exact solution greatly. Numerical examples are presented to show its efficiency.

Mathematics Subject Classification: 35K15, 35C05, 65D99, 65M99

Keywords: Modified homotopy perturbation method; two dimensional sine-Gordon equation; approximate solution; exact solution

1 Introduction

Consider the initial value problem of the two dimensional sine-Gordon(for short tdSG) equation given by

$$u_{tt} - u_{xx} - u_{yy} + m^2 \sin(u) = 0, \tag{1}$$

subject to the initial conditions

$$u(x, y, t_0) = f(x, y),\ u_t(x, y, t_0) = g(x, y), \tag{2}$$

where the subscripts denote the derivation of u with respect to x, y and t. The tdSG equation has been extensively researched since its wide applications and important mathematical properties [1, 8, 10, 12, 13, 15]. Fan et.al. applied the extended tanh method to obtain its explicit solutions [1]. Kaya proposed the modified decomposition method to calculate its explicit and numerical solutions [8, 14]. Different from the existing methods, we will investigate the tdSG equation elegantly by the modified homotopy perturbation method. The homotopy perturbation method(for short HPM) proposed by He has been applied to different linear and nonlinear problems [2, 3, 4, 5, 7]. However, due to the nonlinear term $\sinh(u)$ in (1), the computation cost increases greatly, this leads to lose the efficiency of the HPM. To avoid this disadvantage, we introduce a small parameter and Taylor series expansion to modify the HPM. We call the new scheme as the modified homotopy perturbation method (for short MHPM). The numerical results are presented to show the effectiveness of the MHPM.

The rest of this paper is organized as follows. In Section 2, we introduce the modified homotopy method based on the Taylor series. Then the initial value problems of the tdSG equation are solved by the MHPM in section 3. We end up this paper with some comments.

2 Modified homotopy perturbation method

Based on the homotopy perturbation method [2, 3, 9], we introduce a variable parameter $p \in [0, 1]$ in the two dimensional sine-Gordon equation (1), that is

$$u_{tt} - pu_{xx} - pu_{yy} + m^2 \sin(pu) = 0, \tag{3}$$

subject to the initial conditions

$$u(x, y, t_0) = f(x, y),\ u_t(x, y, t_0) = g(x, y). \tag{4}$$

Obviously, the equation above reduces to a linear equation if $p = 0$, and the equation above is just the original nonlinear one if $p = 1$.

The embedding parameter p is introduced naturally, and unaffected by artificial factors. Furthermore, it can be considered as a small parameter because of $p \in [0, 1]$. By applying the perturbation technique used in [2, 3, 6, 9, 11], we assume that the solution to Eqs.(3) can be expressed in terms of p as follows

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots. \tag{5}$$

Setting $p = 1$ results in the approximate solution of Eqs.(1)

$$u^* = \lim_{p \to 1} u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \cdots. \tag{6}$$

To obtain the approximate solution of Eqs.(3), we consider the Taylor series of $\sin(u)$ in the following form

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots + (-1)^{n-1}\frac{u^{2n-1}}{(2n-1)!} + \cdots. \tag{7}$$

Substituting (5) and (7) into Eqs.(3), and equating the terms of the same power of p, we have

$$\begin{aligned}
p^0: &\quad u_{0tt} = 0,\ u_0(x,y,t_0) = f(x,y),\ u_{0t}(x,y,t_0) = g(x,y)\\
p^1: &\quad u_{1tt} - u_{0xx} - u_{0yy} + m^2 u_0 = 0,\ u_1(x,y,t_0) = 0,\ u_{1t}(x,y,t_0) = 0\\
p^2: &\quad u_{2tt} - u_{1xx} - u_{1yy} + m^2 u_1 = 0,\ u_2(x,y,t_0) = 0,\ u_{2t}(x,y,t_0) = 0\\
p^3: &\quad u_{3tt} - u_{2xx} - u_{2yy} + m^2 u_2 - m^2\frac{u_0^3}{3!} = 0,\ u_2(x,y,t_0) = 0,\ u_{2t}(x,y,t_0) = 0\\
&\quad \cdots\cdots
\end{aligned}$$

Solving these equations by simple integral, it follows that u_0, u_1, u_2, u_3 and so on. Thus, we can obtain the n-order approximation

$$u_{approx} = u_0 + u_1 + u_2 + u_3 \cdots + u_n. \tag{8}$$

3 Numerical results

In this section, we consider two initial value problems related to the tdSG equation by using the modified homotopy perturbation method. The numerical results show the efficiency of this method. All computations are performed by Mathematica software.

We first consider the tdSG equation

$$u_{tt} - u_{xx} - u_{yy} + m^2\sin(u) = 0, \tag{9}$$

subject to the initial conditions

$$\begin{aligned}
u(x,y,0) &= \arccos\left[\frac{\coth^2(K(x+dy))(1+\tanh^4(K(x+dy)))}{2}\right], \qquad (10)\\
u_t(x,y,0) &= \frac{cK\cosh(2K(x+dy))\text{csch}^3(K(x+dy))\text{sech}^3(K(x+dy))}{\sqrt{1-\frac{\coth^4(K(x+dy))(1+\tanh^2(K(x+dy))^4}{4}}}.
\end{aligned}$$

The exact solution of (9) is given by

$$u(x,y,t) = \arccos\left[\frac{\coth^2(K(ct+x+dy))(1+\tanh^4(K(ct+x+dy)))}{2}\right],$$

which represents a breather-kink and antikink transition associated with a double point in the nonlinear spectrum of the sine-Gordon equation [1, 10].

Applying the modified homotopy perturbation method with the initial approximation

$$u_0 = \arccos\left[\frac{\coth^2(K(x+dy))(1+\tanh^4(K(x+dy)))}{2}\right] + \frac{cKt\cosh(2K(x+dy))\mathrm{csch}^3(K(x+dy))\mathrm{sech}^3(K(x+dy))}{\sqrt{1-\frac{\coth^4(K(x+dy))(1+\tanh^2(K(x+dy))^4}{4}}}$$

the approximate solution can be obtained.

To demonstrate the performance of the MHPM, we present the absolute errors $|u_3(0.5,y,t)-u(0.5,y,t)|$ with respect to the third-order approximate solution. Here we have used $m=5$, $\lambda=1$, $K=1$ and $d=\frac{1}{2}\sqrt{-4+4\lambda^2+\frac{m^2}{K^2}}$. From the Table 1, it's obvious that the modified homotopy perturbation method leads to remarkable accuracy of the approximate solution. It's important to note that we use only four terms that is u_0, u_1, u_2 and u_3 to approximate the analytical solution. If we consider more components, the accuracy of the obtained solution will be improved greatly. Moreover, since it does not involve in the complicated calculation of the Adomian polynomials, and the integral associated with $\sin(u)$ that is inevitable in the homotopy perturbation method, we can conclude that this method is more powerful for solving the two dimensional sine-Gordon equation.

We next consider the tdSG equation (9) subject to the following initial conditions

$$u(x,y,0) = \arccos\left[\frac{(1+\coth^4(K(x+dy)))\tanh^2(K(x+dy))}{2}\right], \tag{11}$$

$$u_t(x,y,0) = \frac{cK\cosh(2K(x+dy))\mathrm{csch}^3(K(x+dy))\mathrm{sech}^3(K(x+dy))}{\sqrt{1-\frac{(1+\coth^4(K(x+dy)))^2\tanh^4(K(x+dy)))}{4}}}.$$

The exact solution is given by [1, 10]

$$u(x,y,t) = \arccos\left[\frac{\tanh^2(K(ct+x+dy))(1+\coth^4(K(ct+x+dy)))}{2}\right].$$

Similarly, with the initial approximation

$$u_0 = \arccos\left[\frac{(1+\coth^4(K(x+dy)))\tanh^2(K(x+dy))}{2}\right] + \frac{cKt\cosh(2K(x+dy))\mathrm{csch}^3(K(x+dy))\mathrm{sech}^3(K(x+dy))}{\sqrt{1-\frac{(1+\coth^4(K(x+dy)))^2\tanh^4(K(x+dy)))}{4}}},$$

Table 1 The absolute errors for the approximate solutions of tdSG with the initial conditions (10)

(y,t)	$c = 0.5$	$c = 1$
$(0.01, 0.01)$	0.000095	$3.8188e - 11$
$(0.02, 0.02)$	0.000355	$4.9048e - 10$
$(0.03, 0.03)$	0.000796	$1.9889e - 9$
$(0.04, 0.04)$	0.001817	$4.8980e - 9$
$(0.05, 0.05)$	0.002934	$8.5867e - 9$

Table 2 The absolute errors for the approximate solutions of tdSG with the initial conditions (11)

(y,t)	$c = 0.5$	$c = 1$
$(0.01, 0.01)$	0.000095	$3.8179e - 11$
$(0.02, 0.02)$	0.000355	$4.9048e - 10$
$(0.03, 0.03)$	0.000751	$1.9886e - 9$
$(0.04, 0.04)$	0.001257	$4.8977e - 9$
$(0.05, 0.05)$	0.001854	$8.5857e - 9$

the third-order approximate solution can be derived by the MHPM. The absolute errors $|u_3(0.5, y, t) - u(0.5, y, t)|$ are presented in the Table 2. Again, the approximate solution obtained is in agreement with the exact solution.

4 Conclusion

The initial value problems of two dimensional sine-Gordon equation have been investigated by using the modified homotopy perturbation. The modifications lead to avoid the computation of the Adomian polynomials or the integral of $\sinh(u)$. The numerical results have shown the efficiency of this modified method over the existing methods.

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