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Fixed Point Theorems in Menger Space Using Semi-Compatibility

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Abstract

In this paper we prove a fixed point theorem for six mappings in Menger space using the notion of semi-compatibility.

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1. Introduction

In 1942, K. Menger introduced the notion of probabilistic metric space in which a distribution function was used instead of non-negative real number as value of the metric [4]. The study of these spaces expanded rapidly with the pioneering works of Schweizer and Sklar [8].

In 1972, V. M. Sehgal and A. T. Bharucha-Reid [9] initiated the study of contraction mappings on probabilistic metric (briefly, PM) spaces. Since then there has been a massive growth of fixed point theorems using certain conditions on the mappings or on the space itself.

Sessa [10] introduced weakly commuting maps in metric space. The concept of weakly commuting mappings in probabilistic settings was first studied by Singh and Pant [13, 14]. Jungck [2] introduced the concept of compatible maps. This condition has further been weakened by introducing the notion of weakly-compatible mappings by Jungck and Rhoades [3]. The notion of R-weakly commuting mappings was introduced by Pant [6]. For detailed description of these concepts, we refer to Singh and Tomar [15].

The concept of weakly-compatible mappings is most general as every commuting pair is R-weakly commuting, each pair of R-weakly commuting mappings is compatible and each pair of compatible mappings is weakly compatible but the reverse is not true.

Recently Cho et. al. [1] have introduced the notion of semi-compatible maps in a d-topological space. Singh and Jain [11] have established some fixed point theorems in Menger space using semi-compatibility of the mappings.

In this paper we establish a fixed point theorem for six semi-compatible mappings in Menger space. For further information we refer to Pant and Chauhan [7]. First we recall some definitions and known results in Menger space.

2. Preliminaries

Definition 2.1. A mapping $F: R \to R^+$ is said to be a distribution function if it is non-decreasing, left continuous with $inf\,\{F(t): t \in R\} = 0$ and $sup\,\{F\,(t): t \in R\} = 1$.

We will denote by Δ the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0, & if \quad t \leq 0; \\ 1, & if \quad t > 0. \end{cases}$$

Definition 2.2 [8]. Let X be a non- empty set and Δ denote the set of all distribution functions defined on X. An ordered pair (X, F) is called a probabilistic metric space if F is a mapping from $X \times X$ into Δ satisfying the following conditions:

(i) $F_{x,y}(t) = H\,(t)$ if and only if $x = y$;

(ii) $F_{x,y}(t) = F_{y,x}(t)$;

(iii) $F_{x,y}\,(0) = 0$;

(iv) If $F_{x,z}(t) = 1, F_{z,y}(s) = 1$ then $F_{x,y}(t+s) = 1$ for all x, y, z in X and $t, s \geq 0$.

Every metric space (X, d) can be realized as a probabilistic metric space by taking $F_{x,y}(t) = H(t - d(x,y))$ for all x, y in X.

Definition 2.3 [8]. A mapping $T: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if T satisfies the following conditions:

(i) $T(a,1) = a, T(0,0) = 0$;
(ii) $T(a,b) = T(b,a)$;
(iii) T is continuous;
(iv) $T(a,b) \leq T(c,d)$; whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$;
(v) $T(T(a,b),c) = T(a, T(b,c))$ for all a, b, c in $[0,1]$.

Definition 2.4. A Menger space (X, F, T) is an ordered triad, where T is a continuous t-norm and (X, F) is a probabilistic metric space satisfying the following conditions:

$$F_{x,z}(t+s) \geq T\left(F_{x,y}(t), F_{y,z}(s)\right), \text{ for all } x, y, z \text{ in } X \text{ and } t, s \geq 0.$$

Definition 2.5. Let (X, F, T) be a Menger space.

(i) A sequence $\{x_n\}$ in (X, F, T) is said to converge to a point $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.
(ii) A sequence $\{x_n\}$ is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$.
(iii) A Menger space (X, F, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Definition 2.6 [2]. The self maps A and B of a Menger space (X, F, T) are said to be compatible if

$$F_{ABx_n,BAx_n}(t) \to 1 \text{ for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Bx_n \to x$ for some x in X as n $\to \infty$.

Definition 2.7 [3]. The self maps A and B of a Menger space (X, F, T) are said to be weakly compatible if they commute at their coincidence points, i.e; if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Proposition 2.8 [12]. *If the self maps A and B of a Menger space (X, F, T) are compatible then they are weakly compatible.*

Definition 2.9 [7, 11]. A pair (A, B) of self maps of a Menger space (X, F, T) is said to be semi-compatible if

$F_{ABx_n,Bx}$ (t) $\to 1$ for all $t > 0$,

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$. It follows that if (A, B) is semi-compatible and $Ax = Bx$ then $ABx = BAx$. Thus if the pair (A, B) is semi-compatible then it is weakly compatible. The converse is not true as seen in example 2.11.

Proposition 2.10. *Let A and B be self maps on a Menger space (X, F, T). If B is continuous then (A, B) is semi-compatible if and only if (A, B) is compatible.*

In the following example we shall show that a pair of semi-compatible self maps (A, B) need not be compatible and semi-compatibility of a pair of self maps implies weak compatibility.

Example 2.11. Let $X = [0, 2]$ with the usual metric d, i.e; $d(x, y) = |x - y|$ for all x, y in X. Define $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all x, y in X and for all $t > 0$.

We define self maps A and B as follows:

$$A(x) = \begin{cases} x, & if \quad 0 \le x < 1; \\ 2, & if \quad 1 \le x \le 2. \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 2 - x, & if \quad 0 \le x < 1; \\ 2, & if \quad 1 \le x \le 2. \end{cases}$$

Taking $x_n = 1 - \frac{1}{n}$. We get $Ax_n = 1 - \frac{1}{n}$, $Bx_n = 1 + \frac{1}{n}$. Thus $Ax_n \to 1$, $Bx_n \to 1$. Hence $x = 1$. Further $ABx_n = 2$, $BAx_n = 1 + \frac{1}{n}$.

Now; $\lim_{n\to\infty} F_{ABx_n,BAx_n}$ (t) $= \lim_{n\to\infty} F_{2,1+\frac{1}{n}}$ (t) $= \frac{t}{t+1} < 1$, for all t > 0.

Hence A and B are not compatible. Also,

$\lim_{n\to\infty} F_{ABx_n,Bx}(\text{t}) = \lim_{n\to\infty} F_{2,2}(\text{t}) = 1$.

So it is clear that A, B are semi-compatible.

And $\lim_{n\to\infty} F_{BAx_n,Ax}(\text{t}) = \lim_{n\to\infty} F_{1+\frac{1}{n},2}(\text{t}) = \frac{t}{t+1} < 1$, for all $t > 0$.

So it is clear that B, A are not semi-compatible. Now we will show that the semi-compatible pair (A, B) is weakly compatible also.

Coincidence points of A and B are in [1, 2]. Now for any $x \in$ [1, 2]. $Ax = Bx = 2$ and $AB(x) = A(2) = 2 = B(2) = BA(x)$. Thus (A, B) is weakly compatible.

Lemma 2.12 [5]. *Let $\{x_n\}$ be a sequence in a Menger space (X, F, T) with continuous t-norm T and $T(a, a) \ge a$. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x_n, x_{n+1}}(kt) \ge F_{x_{n-1}, x_n}(t)$$

for all t > 0 and n =1, 2, 3, ... then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.13 [5]. *Let (X, F, T) be a Menger space. If there exists $k \in (0, 1)$ such that*

$F_{x,y}(kt) \geq F_{x,y}(t)$
for all $x, y \in X$ *and* $t > 0$ *then* $x = y$.

3. Main Results

Theorem 3.1. *Let* A, B, S, T, L *and* M *be self maps on a complete Menger space* (X, F, T) *with continuous* t*-norm* T *defined by* $T\,(a, b) = min\,\{a, b\}$ *for* $a, b \in [0, 1]$ *and satisfy the following:*

(a) $AB(X) \subset M(X)$ *and* $ST(X) \subset L(X)$*;*
(b) $M(X)$ *and* $L(X)$ *are complete subspace of* X*;*
(c) *Either* AB *or* ST *is continuous;*
(d) (AB, L) *is semi-compatible and* (ST, M) *is weakly compatible;*
(e) *For all* $x, y \in X$ *,* $k \in (0, 1), t > 0,$

$$F^3_{ABx,STy}(kt) \geq min \begin{Bmatrix} F^3_{Lx,My}(t), F^3_{ABx,Lx}(t), F^3_{STy,My}(t), F_{ABx,My}(2t), \\ F_{STy,Lx}(2t), F^2_{STy,My}(t) \end{Bmatrix}$$

Then AB, ST, L *and* M *have a unique common fixed point in* X.

Proof: Let $x_0 \in X$ be any arbitrary point. Since $AB\,(X) \subset M\,(X)$ and $ST\,(X) \subset L\,(X)$, there exist $x_1, x_2 \in X$ such that $ABx_0 = Mx_1$ and $STx_1 = Lx_2$. Inductively, we construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n-1} = MX_{2n-1} = ABx_{2n-2}$ and $y_{2n} = Lx_{2n} = STx_{2n-1}$ for $n = 1,2,3$ … …. .
Step (i): Now using (e) with $x = x_{2n}$, $y = x_{2n+1}$, we have

$$F^3_{y_{2n+1},y_{2n+2}}(kt) = F^3_{ABx_{2n},STx_{2n+1}}(kt)$$
$$\geq$$
$$\min \begin{Bmatrix} F^3_{Lx_{2n},Mx_{2n+1}}(\mathrm{t}), F^3_{ABx_{2n},Lx_{2n}}(\mathrm{t}), F^3_{STx_{2n+1},Mx_{2n+1}}(\mathrm{t}), F_{ABx_{2n},Mx_{2n+1}}(2\mathrm{t}), \\ F_{STx_{2n+1},Lx_{2n}}(2\mathrm{t}), F^2_{STx_{2n+1},Mx_{2n+1}}(\mathrm{t}) \end{Bmatrix}$$
$$\geq \min \begin{Bmatrix} F^3_{y_{2n},y_{2n+1}}(\mathrm{t}), F^3_{y_{2n},y_{2n+1}}(\mathrm{t}), F^3_{y_{2n+2},y_{2n+1}}(\mathrm{t}), F_{y_{2n+1},y_{2n+1}}(2\mathrm{t}), \\ F_{y_{2n+2},y_{2n}}(2\mathrm{t}), F^2_{y_{2n+2},y_{2n+1}}(\mathrm{t}) \end{Bmatrix}$$
$$\geq \min \begin{Bmatrix} F^3_{y_{2n},y_{2n+1}}(\mathrm{t}), F^3_{y_{2n+2},y_{2n+1}}(\mathrm{t}), \\ \min\{F_{y_{2n+2},y_{2n+1}}(\mathrm{t}), F_{y_{2n+1},y_{2n}}(\mathrm{t})\}, F^2_{y_{2n+2},y_{2n+1}}(\mathrm{t}) \end{Bmatrix}$$

$F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(t)$.
Similarly we can have
$F_{y_{2n},y_{2n+1}}(kt) \geq F_{y_{2n-1},y_{2n}}(t)$
Therefore, for all n we have:
$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t)$
Hence by Lemma 2.12, $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore $\{y_n\}$ converges to $z \in X$. Thus its subsequences also converge to z.

Case (I): Let AB be continuous, then $AB(AB)x_{2n} \to ABz$ and $(AB)Lx_{2n} \to ABz$. As (AB, L) is semi-compatible, $(AB)Lx_{2n} \to Lz$.
By uniqueness of limit in Menger space, we obtain $ABz = Lz$.
Step (ii): Putting $x = z$ and $y = x_{2n+1}$ in (e), we have

$$F^3_{ABz,STx_{2n+1}}(kt) \geq \min\left\{\begin{matrix} F^3_{Lz,Mx_{2n+1}}(\mathrm{t}), F^3_{ABz,Lz}(\mathrm{t}), F^3_{STx_{2n+1},Mx_{2n+1}}(\mathrm{t}), F_{ABz,Mx_{2n+1}}(2\mathrm{t}), \\ F_{STx_{2n+1},Lz}(2\mathrm{t}), F^2_{STx_{2n+1},Mx_{2n+1}}(\mathrm{t}) \end{matrix}\right\}$$

Taking limit, we get

$$\begin{aligned} F^3_{ABz,z}(kt) &\geq \min\left\{F^3_{Lz,z}(\mathrm{t}), F^3_{ABz,Lz}(\mathrm{t}), F^3_{z,z}(\mathrm{t}), F_{ABz,z}(2\mathrm{t}), F_{z,Lz}(2\mathrm{t}), F^2_{z,z}(\mathrm{t})\right\} \\ &\geq \min\left\{F^3_{ABz,z}(\mathrm{t}), 1, 1, F_{ABz,z}(2\mathrm{t}), F_{z,ABz}(2\mathrm{t}), 1\right\} \\ &\geq F^3_{ABz,z}(t). \end{aligned}$$

Hence $ABz = z$.
Step (iii): As $AB(X) \subset M(X)$, there exists a point $v \in X$ such that $z = ABz = Mv$.
By taking $x = z$ and $y = v$ in (e), we have

$$F^3_{ABz,STv}(kt) \geq \min\left\{F^3_{Lz,Mv}(\mathrm{t}), F^3_{ABz,z}(\mathrm{t}), F^3_{STv,Mv}(\mathrm{t}), F_{ABz,Mv}(2\mathrm{t}), F_{STv,Lz}(2\mathrm{t}), F^2_{STv,Mv}(\mathrm{t})\right\}$$

Taking limit, we get

$$\begin{aligned} F^3_{z,STv}(kt) &\geq \min\left\{F^3_{ABz,ABz}(\mathrm{t}), 1, F^3_{STv,Mv}(\mathrm{t}), F_{Mv,Mv}(2\mathrm{t}), F_{STv,ABz}(2\mathrm{t}), F^2_{STv,Mv}(\mathrm{t})\right\} \\ &\geq \min\left\{1, 1, F^3_{STv,z}(\mathrm{t}), 1, F_{STv,z}(2\mathrm{t}), F^2_{STv,z}(\mathrm{t})\right\} \end{aligned}$$

$$F^3_{z,STv}(kt) \geq F^3_{z,STv}(t)$$

Hence $z = STv = Mv$.
Step (iv): As (ST, M) is weakly compatible, $M(STv) = ST(Mv)$. Hence $Mz = STz$.
By taking $x = z$ and $y = z$ in (e), we get

$$\begin{aligned} F^3_{ABz,STz}(kt) &\geq \min\left\{\begin{matrix} F^3_{Lz,Mz}(\mathrm{t}), F^3_{ABz,Lz}(\mathrm{t}), F^3_{STz,Mz}(\mathrm{t}), F_{ABz,Mz}(2\mathrm{t}), \\ F_{STz,Lz}(2\mathrm{t}), F^2_{STz,Mz}(\mathrm{t}) \end{matrix}\right\} \\ &\geq \min\left\{\begin{matrix} F^3_{ABz,STz}(\mathrm{t}), F^3_{Lz,Lz}(\mathrm{t}), F^3_{Mz,Mz}(\mathrm{t}), F_{ABz,STz}(2\mathrm{t}), \\ F_{STz,ABz}(2\mathrm{t}), F^2_{Mz,Mz}(\mathrm{t}) \end{matrix}\right\} \\ &\geq \min\left\{F^3_{ABz,STz}(\mathrm{t}), 1, 1, F_{ABz,STz}(2\mathrm{t}), 1\right\} \\ &\geq F^3_{ABz,STz}(t) \end{aligned}$$

Hence; $ABz = STz$.
Now, combine all the results, we get $z = ABz = STz = Lz = Mz$. Thus z is the common fixed point of AB, ST, L and M.
Case II: If ST is continuous, the proof follows by case I.

Uniqueness: Let $w (w \neq z)$ be another common fixed point of AB, ST, L and M, then $w = ABw = STw = Lw = Mw$.
By taking $x = z$ and $y = w$ in (e), we get

$$F^3_{ABz,STw}(kt) \geq \min \begin{Bmatrix} F^3_{Lz,Mw}(\mathrm{t}), F^3_{ABz,Lz}(\mathrm{t}), F^3_{STw,Mw}(\mathrm{t}), F_{ABz,Mw}(2\mathrm{t}), \\ F_{STw,Lz}(2\mathrm{t}), F^2_{STw,Mw}(\mathrm{t}) \end{Bmatrix}$$

$$F^3_{z,w}(kt) \geq \min\left\{F^3_{z,w}(\mathrm{t}), F^3_{z,z}(\mathrm{t}), F^3_{w,w}(\mathrm{t}), F_{z,w}(2\mathrm{t}), F_{w,z}(2\mathrm{t}), F^2_{w,w}(\mathrm{t})\right\}$$
$$\geq F^3_{z,w}\ (t).$$

Hence $z = w$ for all $x, y \in X$ and $t > 0$. Therefore z is the unique common fixed point of AB, ST, L and M.

On taking $B = T = I$ (identity maps) in Theorem 3.1, then we have the following:

Corollary 3.2. *Let A, S, L and M be self maps on a complete Menger space (X, F, T) with continuous t-norm T defined by $T(a, b) = min\,\{a, b\}$ for $a, b \in [0, 1]$ and satisfy the following:*

(a) $A(X) \subset M(X)$ *and* $S(X) \subset L(X)$*;*

(b) $M(X)$ *and* $L(X)$ *are complete subspace of X;*

(c) *Either A or S is continuous;*

(d) (A, L) *is semi-compatible and* (S, M) *is weakly compatible;*

(e) *For all* $x, y \in X, k \in (0, 1), t > 0$

$$F^3_{Ax,Sy}(kt) \geq min \begin{Bmatrix} F^3_{Lx,My}(t), F^3_{Ax,Lx}(t), F^3_{Sy,My}(t), F_{Ax,My}(2t), \\ F_{Sy,Lx}(2t), F^2_{Sy,My}(t) \end{Bmatrix}$$

Then A, S, L and M have a unique common fixed point in X.

If we take $A = S, L = M$ and $B = T = I$ (identity mappings) in Theorem 3.1 then we have the following:

Corollary 3.3. *Let A and L be self maps on a complete Menger space (X, F, T) with continuous t-norm T defined by $T(a, b) = min\,\{a, b\}$ for $a, b \in [0, 1]$ and satisfy the following:*

(a) $A(X) \subset L(X)$*;*

(b) $L(X)$ *is complete subspace of X;*

(c) *L is continuous;*

(d) (A, L) *is semi-compatible;*

(e) *For all* $x, y \in X, k \in (0, 1), t > 0,$

$$F^3_{Ax,Ay}(kt) \ \geq min \begin{Bmatrix} F^3_{Lx,Ly}(t), F^3_{Ax,Lx}(t), F^3_{Ay,Ly}(t), F_{Ax,Ly}(2t), \\ F_{Ay,Lx}(2t),\ F^2_{Ay,Ly}(t) \end{Bmatrix}$$

Then A and L have a unique common fixed point in X.

□

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