Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 19, 901 - 910

Existence and Controllability Result for an Evolution Fractional Integrodifferential Systems

Mabrouk Bragdi[†] **and Mohammed Hazi** [‡]

[†]Department of mathematics, Larbi Ben M'Hidi University
04000, OEB, Algeria
bravdi@yahoo.com

[‡]Department of mathematics, École Normale Supèrieure
16050-Kouba, Algiers, Algeria
hazi@ens-kouba.dz

Abstract

In this paper we study the existence and the controllability of fractional evolution integrodifferential systems in Banach spaces. The results are obtained with the help of resolvent operators, fractional calculus and a fixed point analysis approach.

Mathematics Subject Classification: 93B05, 26A33, 34G20

Keywords: Controllability, Mild solution, Resolvent operators, Fractional calculus, Banach fixed point theorem

1 Introduction

In the recent years, the studies of fractional differential equations and control problems have attracted the attention of many mathematician and physicists. A large amount of literature developed concerning fractional differential equations [2, 4, 7, 8], and their controllability [2] to investigated various scientific models. Motivated by the fact that many partial fractional differential and integrodifferential equations can be converted into fractional equations in some Banach spaces [1, 5, 13], we feel that there is a real need to discuss the controllability problem of fractional order systems in abstract spaces [13].

In this paper, we study the existence and controllability result with the help of resolvent operators.

This work is organized as follows: Section 2 gives the preliminaries for the paper. In Sections 3, the existence result of the fractional integrodifferential systems with the help of resolvent operators is studied. In Sections 4 the controllability of the fractional integrodifferential control systems is investigated via the Banach fixed point theorem. The last Section is devoted to an illustrated example.

2 Preliminaries

In this section we give some basic definitions and results about fractional calculus and resolvent operators.

Definition 2.1 ([1, 9]) A real function $f(t)$ is said to be in the space $C_\alpha, \alpha \in R$ if there exists a real number $p > \alpha$ such that $f(t) = t^p g(t)$, where $g \in C[0, \infty]$ and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha, m \in N$.

Definition 2.2 ([1, 9]) The Riemann-Liouville fractional integral operator of order-$\beta > 0$ of function $f \in C_\alpha,\ \alpha \geq -1$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds.$$

Definition 2.3 ([1, 9]) if the function $f \in C_{-1}^m$ and m is a positive integer, then we can define the fractional derivative of $f(t)$ in the Caputo sense as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds,\ m-1 < \alpha \leq m.$$

If $0 < \alpha \leq 1$, then $\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds$, where $f'(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X.

Let X be a Banach space and the symbol $\|\cdot\|$ denotes the norm of all spaces and bounded linear operators considered in this work, it is also denotes the sup-norm of any bounded continuous function.

Definition 2.4 ([10]) A two parameter family of bounded linear operators $U(t, s)$, $0 \leq s \leq t \leq T$, on X is called an evolution system if the following two conditions are satisfied.

(i) $U(t, t) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$,

(ii) $(t, s) \to U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

In the section 3, we consider the nonautonomous semilinear evolution fractional integrodifferential system of the form

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = A(t)x(t) + f\left(t, x(t), \int_0^t h(t,s,x(s))ds\right), \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where $t \in J := [0,b]$, $0 < \alpha < 1$, the state $x(\cdot)$ takes values in the Banach space X, $x_0 \in X$. Here the family $\{A(t)\}_{t\in J}$ is a closed linear operator on X with dense domain $D(A)$ which is independent of t and generates an evolution operators in the Banach space X. The nonlinear operators $h : \Delta \times X \to X$ and $f : J \times X \times X \to X$ are given function and $\Delta = \{(t,s) \mid 0 \le s \le t \le b\}$.

In the section 4, we consider the semilinear evolution fractional integrodifferential control system of the form

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = A(t)x(t) + f\left(t, x(t), \int_0^t h(t,s,x(s))ds\right) + Bu(t), \tag{2.3}$$

$$x(0) = x_0, \tag{2.4}$$

where the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space and B is a bounded linear operator from U to X.

Definition 2.5 ([11, 12]) A resolvent operator for equations (2.1)–(2.4) is a bounded operator valued function $R(t,s) \in B(X)$, $0 \le s \le t \le b$, the space of bounded linear operators on X, having the following proprieties.

(i) $R(t,s)$ is strongly continuous in s and t, $R(t,t) = I$, $0 \le t \le b$, $\|R(t,s)\| \le Me^{\beta(t-s)}$ for som constants M and β.

(ii) $R(t,s)Y \subset Y$, $R(t,s)$ is strongly continuous in s and t on Y.

(iii) For $x \in X$, $R(t,s)x$ is continuously differentiable in $s \in [0,b]$ and

$$\frac{\partial R}{\partial s}(t,s)\,x = -R(t,s)\,A(s)\,x.$$

(iv) For $x \in X$ and $s \in [0,b]$, $R(t,s)x$ is continuously differentiable in $t \in [s,b]$ and

$$\frac{\partial R}{\partial t}(t,s)\,x = A(t)\,R(t,s)\,x,$$

with $\frac{\partial R}{\partial s}(t,s)\,x$ and $\frac{\partial R}{\partial t}(t,s)\,x$ strongly continuous on $0 \le s \le t \le b$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space. It will not, however, be an evolution operator because it will not satisfy an evolution or semigroup property. Because a number of results follow directly from the definition of the resolvent operator.

The following definition is similar to the concept defined in [1, 10, 11]

Definition 2.6 A continuous function $x(\cdot) : J \to X$ is said to be a mild solution of the problem (2.1)–(2.2) on J, if it satisfies the following integral equation

$$\begin{aligned} x(t) &= R(t,0)\,x_0 \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R(t,s)\, f\left(s, x(s), \int_0^s h(s,\tau,x(\tau))\,d\tau\right) ds. \end{aligned}$$

Definition 2.7 ([1, 3]) The system (2.3)–(2.4) is said to be controllable on the interval J, if for every $x_0,\ x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of (2.3)–(2.4) corresponding to u satisfies $x(b) = x_1$.

We shall make the following hypotheses

(H1) $f : J \times X \times X \to X$ is continuous and there exists constants $M_1, M_2 > 0$, such that for all $x_i, y_i \in X, i = 1, 2$ we have

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le M_1 \left[\|x_1 - x_2\| + \|y_1 - y_2\|\right],$$

and

$$M_2 = \max_{t \in J} \|f(t,0,0)\|.$$

(H2) $h : \Delta \times X \to X$ is continuous and there exists constants $L_1, L_2 > 0$, such that for all $x_1, x_2 \in X$ we have

$$\|h(t,s,x_1) - h(t,s,x_2)\| \le L_1 \|x_1 - x_2\|,$$

and

$$L_2 = \max_{t,s \in \Delta} \|h(t,s,0)\|.$$

(H3) Denote $M = \max_{t,s \in J} \|R(t,s)\|$ for $s < t$.

(H4) Let $M\|x_0\| + \gamma M N_1 \le n$ for some $n > 0$, where $N_1 = M_1(n + nL_1 b + L_2 b) + M_2$, $\gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}$ and $p_1 = \gamma M\left[M_1 + M_1 L_1 b\right]$ be such that $0 \le p_1 < 1$.

Under these assumptions, we can prove the existence of mild solutions.

3 Existence result

Theorem 3.1 *Let assumptions* (H1)–(H4) *be satisfied. Then for every* $x_0 \in X$*, problem* (2.1)–(2.2) *has a mild solution on* J*.*

Proof As in [11], we let $x_0 \in X$ be fixed and let $Z = C(J, X)$ be the Banach space of all continuous functions from J into X with sup-norm and $Z_n = \{\, x \mid x \in Z,\ x(0) = x_0,\ \|x(t)\| \le n, \text{ for t} \in \text{J} \,\}$. Define an operator $\Phi : Z_n \to Z_n$ by

$$\Phi x(t) = R(t,0)\,x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R(t,s)\, f(s, x(s), Hx(s))\, ds,$$

where $Hx(s) = \int_0^s h(s, \tau, x(\tau))\, d\tau$.
First we show that Φ maps Z_n into it self. For $x \in Z_n$

$$\begin{aligned}
\| \Phi x(t) \| &\le \| R(t,0)\,x_0 \| \\
&\quad + \frac{1}{\Gamma(\alpha)} \| \int_0^t (t-s)^{\alpha-1} R(t,s)\, f(s, x(s), Hx(s))\, ds \| \\
&\le \| R(t,0)\,x_0 \| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| R(t,s) \| \\
&\quad \times \Big[\| f(s, x(s), Hx(s)) - f(s,0,0) \| + \| f(s,0,0) \| \Big] ds \\
&\le M \|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M \left[M_1 (n + nL_1 b + L_2 b) + M_2 \right] \\
&\le M \|x_0\| + \gamma M N_1 \\
&\le n.
\end{aligned}$$

Therefor Φ maps Z_n into it self. Now for $x, y \in Z_n$ we have

$$\begin{aligned}
\| \Phi x(t) - \Phi y(t) \| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| R(t,s) \| \, \| f(s, x(s), Hx(s)) \\
&\quad - f(s, y(s), Hy(s)) \| \, ds \\
&\le \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \left[M_1 + M_1 L_1 b \right] \|x - y\| \\
&\le \frac{\mathrm{M}\mathrm{b}^\alpha}{\Gamma(\alpha+1)} \left[M_1 + M_1 L_1 b \right] \|x - y\|,
\end{aligned}$$

Thus

$$\| \Phi x - \Phi y \| \le \gamma M \left[M_1 + M_1 L_1 b \right] \|x - y\| = p_1 \|x - y\|.$$

which implies that Φ is a contraction on Z_n by assumption (H4).

4 Controllability result

For all $x_0 \in X$ and admissible control $u \in L^2(J,U)$, system (2.3)–(2.4) admits a mild solution give by

$$\begin{aligned} x_u(t) &= R(t,0)\,x_0 \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} R(t,s)\Big[Bu(s)+f(s,x(s),Hx(s))\Big]\,ds. \end{aligned}$$

To prove the controllability result, we further consider the following additional conditions.

(H5) The bounded linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} R(b,s)\,Bu(s)\,ds,$$

has an induced inverse operator $\tilde{W}^{-1}$ which takes values in $L^2(J,U)/\ker W$ and there exists constants $K_1, K_2 > 0$, such that $\| B \| \le K_1$ and $\| \tilde{W}^{-1} \| \le K_2$

(H6) Let $M\|x_0\| + \gamma M K_1 K_2 [\|x_1\| + M\|x_0\| + \gamma M N_2] + \gamma M N_2 \le k$ for some $k > 0$, where $N_2 = M_1(k + kL_1 b + L_2 b) + M_2$ and $\gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}$.

(H7) $p_2 = \gamma M M_1 [MK_1K_2\gamma + 1][1 + L_1 b]$ be such that $0 \le p_2 < 1$.

Theorem 4.1 *If the hypotheses* (H1)–(H3) *and* (H5)–(H7) *are satisfied, then the control system* (2.3)–(2.4) *is controllable on* J.

Proof Let $Z_k = \{\, x \mid x \in Z,\ x(0) = x_0,\ \|x(t)\| \le k, \text{ for t} \in \text{J} \,\}$. Define an operator $\Omega : Z_k \to Z_k$ by

$$\begin{aligned} \Omega x_u(t) &= R(t,0)\,x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} R(t,\eta)\,B\tilde{W}^{-1}\Big[x_1 - R(b,0)\,x_0 \\ &-\frac{1}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1} R(b,s)\,f(s,x(s),Hx(s))\,ds\Big](\eta)\,d\eta \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} R(t,s)\,f(s,x(s),Hx(s))\,ds. \end{aligned}$$

Using the hypothesis (H5), for an arbitrary function $x(\cdot)$ choose the control

$$\begin{aligned} u(t) &= \tilde{W}^{-1}\Big[x_1 - R(b,0)\,x_0 \\ &-\frac{1}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1} R(b,s)\,f(s,x(s),Hx(s))\,ds\Big](t). \end{aligned}$$

Using this control we shall show that the operator Ω has a fixed point. This fixed point is then a solution of the control problem (2.3)–(2.4). Clearly $\Omega x(b) = x_1$, which means that the control u steers the system (2.3)–(2.4) from initial state x_0 to x_1 in time b provided we can obtain a fixed point of the operator Ω. As in [1, 11] we first show that Ω maps Z_k into itself.

$$\begin{aligned}
\| \Omega x_u(t) \| &\leq \| R(t,0)x_0 \| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} \| R(t,\eta) \| \| B \| \| \tilde{W}^{-1} \| \\
&\quad \times \Big[\| x_1 \| + \| R(b,0)x_0 \| + \frac{1}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1} \| R(b,s) \| \\
&\quad \times \Big[\| f(s,x(s),Hx(s)) - f(s,0,0) \| + \| f(s,0,0) \| \Big] ds \Big] d\eta \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \| R(t,s) \| \Big[\| f(s,x(s),Hx(s)) \\
&\quad - f(s,0,0) \| + \| f(s,0,0) \| \Big] ds \\
&\leq M \| x_0 \| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} MK_1K_2 \Big[\| x_1 \| + M \| x_0 \| \\
&\quad + \frac{b^\alpha}{\Gamma(\alpha+1)} M \left[M_1(k + kL_1b + L_2b) + M_2 \right] \Big] d\eta \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} M \left[M_1(k + kL_1b + L_2b) + M_2 \right] ds \\
&\leq M\|x_0\| + \gamma MK_1K_2 \left[\|x_1\| + M\|x_0\| + \gamma MN_2 \right] + \gamma MN_2 \\
&\leq k.
\end{aligned}$$

Thus, Ω maps Z_k into itself. Now, for $x, y \in Z_k$, we have

$$\begin{aligned}
\| \Omega x(t) - \Omega y(t) \| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1} \| R(t,\eta) B\tilde{W}^{-1} \| \\
&\quad \times \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} \| R(b,s) \Big[f(s,x(s),Hx(s)) \\
&\quad - f(s,y(s),Hy(s)) \Big] \| ds d\eta \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \| R(t,s) \Big[f(s,x(s),Hx(s)) \\
&\quad - f(s,y(s),Hy(s)) \Big] \| ds \\
&\leq \frac{b^\alpha}{\Gamma(\alpha+1)} MK_1K_2 \frac{b^\alpha}{\Gamma(\alpha+1)} M \left[M_1 + M_1L_1b \right] \|x-y\| \\
&\quad + \frac{b^\alpha}{\Gamma(\alpha+1)} M \left[M_1 + M_1L_1b \right] \|x-y\| \\
&\leq \gamma MM_1 \left[MK_1K_2\gamma + 1 \right] \left[1 + L_1b \right] \|x-y\|,
\end{aligned}$$

Thus

$$\|\Omega x - \Omega y\| \leq p_2 \|x - y\| .$$

From assumption (H7), we find that Ω is a contraction mapping on Z_k and hence there exists a unique fixed point $x \in Z_k$. Any fixed point of Φ is a Mild solution of (2.3)–(2.4) which satisfies $x(b) = x_1$. Thus, system (2.3)–(2.4) is controllable on J.

5 Example

In this section we present an example to illustrate our main results. Let us consider the following nonlinear partial integrodifferential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} z(t,y) = a(t,y)\frac{\partial^2}{\partial t^2} z(t,y) + \mu(t,y) + \mu_1\left(t, z(t,y), \int_0^t \mu_2(t,s,z(t,y))\, ds\right), \tag{5.1}$$

$$z(0,y) = z_0(y), \quad 0 < y < 1, \tag{5.2}$$

$$z(t,0) = z(t,1), \quad t \in J = [0,1], \tag{5.3}$$

where $0 < \alpha < 1$, a and $\mu : J \times [0,1] \to [0,1]$ are continuous functions.
Assume that nonlinear functions μ_1, μ_2 satisfy the following Lipschitz conditions

$$\|\mu_1(t, v_1, w_1) - \mu_1(t, v_2, w_2)\| \leq W_1 [\|v_1 - v_2\| + \|w_1 - w_2\|],$$

$$\|\mu_2(t, s, w_1) - \mu_2(t, s, w_2)\| \leq W_2 \|v_1 - v_2\| .$$

where $W_1, W_2 > 0$, $v_i, w_i \in X, i = 1,2$.

Now, let us take $X = U = L^2[0,1]$, and define $A(t) : X \to X$ by

$$(A(t)w)(y) = a(t,y)w'',$$

with domain $D(A) = \{ w \in X \mid w, w''$ are absolutely continuous, $w'' \in X, w(0) = w(1) = 0 \}$, generates an evolution system (see [11, 12]), such that $R(t,s) \leq k$, $k > 0$ for $s < t$.

Let $Bu : J \to X$ be defined by

$$(Bu)(t)(y) = \mu(t,y), \quad y \in (0,1).$$

Assume that the linear operator W is given by

$$(Wu)(y) = \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} R(b,s)\, \mu(s,y)\, ds, \quad y \in (0,1),$$

has a bonded invertible operator $\tilde{W}^{-1}$ in $L^2(J,U)/\ker W$.

With the choice of A, B, h and f, we see that Eqs. (2.3)–(2.4) is the abstract formulation of Eqs. (5.1)–(5.3).

Further, all the conditions states in theorem 4.1 are satisfied. Hence, system (5.1)–(5.3) is controllable on J.

References

[1] K. Balachandran and J.Y. Park, Controllability of fractional integrodifferential systems in Banach spaces, *Nonlinear Analysis: Hybrid Systems,* (2009), doi:10.1016/j.nahs.2009.01.014.

[2] B. Bonilla, M. Rivero, L. Rodriguez-Germa and J.J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, *Applied Mathematics and Computation,* **187** (2007), 79–88.

[3] Y.Q. Chen, H.S. Ahu and D. Xue, Robust controllability of interval fractional order linear time invariant systems, *Signal Processing,* **86** (2006), 2794–2802.

[4] D. Delbosco and L. Rodino, Existence and uniqueness for a fractional differential equation, *Journal of Mathematical Analysis and Applications,* **204** (1996), 609–625.

[5] M.A.A. El-Sayeed, Fractional order diffusion wave equation, *International Journal of Theoretical Physics,* **35** (1966), 311–322.

[6] O.K. Jaradat, A. Al-Omari and S. Momani, Existence of the mild solution for fractional semilinear initial value problem, *Nonlinear Analysis,* **69** (2008), 3153–3159.

[7] V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Analysis,* **69** (2008), 3337–3343.

[8] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Analysis,* **69** (2008), 2677–2682.

[9] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York, 1993.

[10] A. Pazy, *Semigroup of linear operators and applications to partial differential equations,* Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[11] R. Sakthivel, S.M. Anthoni and J.H. Kim, Existance and controllability result for semilinear evolution integrodifferential systems, *Mathematical and Computer Modelling,* **41** (2005), 1005-1011.

[12] R. Sakthivel, Q.H. Choi and S.M. Anthoni, Controllability of nonlinear neutral evolution integrodifferential systems, *J. Math. Anal. Appl.,* **275** (2002), 402-417.

[13] Z. Tai and X. Wang, Controllability of fractional–order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces, *Appl. Math. Lett.,* (2009), doi:10.1016/j.aml.2009.06.017.

Received: August, 2009