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Euler's Phi Function and Graph Labeling

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Abstract

A graph $G(V, E)$ with vertex set V is said to have a prime labeling if there exist a bijection $f : V(G) \to \{1, 2, \ldots, |V|\}$ such that for each edge $xy \in E(G)$, $f(x)$ and $f(y)$ are relatively prime. It becomes an interesting problem to investigate the maximum possible edges that can be constructed in a graph with n vertices having prime labeling property. We interpret this problem in number theory to find total number of relatively prime pair of integers in the set $\{1, 2, 3, \ldots, n\}$ and prove an interesting result connecting with Euler's phi function $\phi(n)$. Further an algorithmic approach is established for prime labeling for the class of maximal planar graph Pl_n using the number theory techniques.

Mathematics Subject Classification: 05C78

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1 Introduction

In this paper we consider only simple graph without self loops and multiple edges. A Graph $G(V, E)$ is connected if every pair of vertices is joined by a path. A graph which is not connected is called disconnected graph. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of Mathematical Models from a broad range of applications. The graph labeling problem that appears in graph theory has a fast development recently. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. The notion of a prime labeling originated with Entringer and was introduced in a paper in Tout, Dabboucy and Howalla [2].

A graph with vertex set V is said to have a prime labeling if there exist a bijection $f : V(G) \to \{1, 2, \ldots, |V|\}$ such that for each edge $xy \in E(T)$, $f(x)$ and $f(y)$ are relatively prime. In other words the Greatest Common devisor of $f(x)$ and $f(y)$ denoted by g.c.d of $(f(x), f(y)) = 1$.

Let n be a positive integer. The **Euler's phi function** $\phi(n)$ denotes the number of positive integer's $\leq n$ and relatively prime to n.

For example, $\phi(1) = 1$; $\phi(9) = 6$; $\phi(12) = 4$

A positive integer p is a prime if and only if $\phi(p) = p - 1$.

1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the canonical decomposition of a positive integer n. Then $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$

2. $\phi(n)$ is even for $n \geq 3$.

3. Let n be a positive integer. Then $\sum_{d/n} \phi(d) = n$.

Motivation

A (p, q) graph $G = (V, E)$ with p vertices and q edges is called **total edge magic** if there is a bijection function $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ such that for any edge uv in E we have a constant k with $f(u)+f(v)+f(uv) = k$. A total edge-magic graph is called **super edge-magic** if $f(V(G)) = \{1, 2, \ldots, p\}$. A graph $G(p, q)$ with p vertices and q edges is called total edge bimagic if there exists a bijection $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ such that for any edge $uv \in E$, we have two constants k_1 and k_2 with $f(u) + f(v) + f(uv) = k_1$ or k_2.

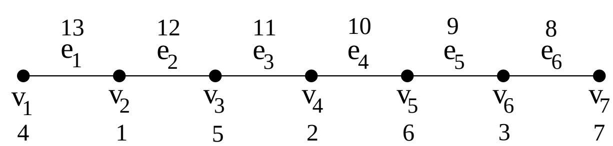

Figure 1: Edge magic labeling for the path graph P_7

The path graph P_n $(n \geq 2)$ having exactly two pendent vertices is edge magic with constant for any edge e,

$$\lambda(e) = \begin{cases} (5n+2)/2 & \text{if } n \cong 0 \mod 2 \\ (5n+1)/2 & \text{if } n \cong 1 \mod 2 \end{cases}$$

The star $K_{1,n-1}$ is edge magic with Common count for every edge e, $\lambda(e) = 2n + 2$.

The notion of edge magic labeling in trees leads to the conclusion that, we can express a particular constant k in $n-1$ ways as a sum of three distinct positive integers from the set $\{1, 2, \ldots, 2n-1\}$. Similarly edge bimagic total labeling

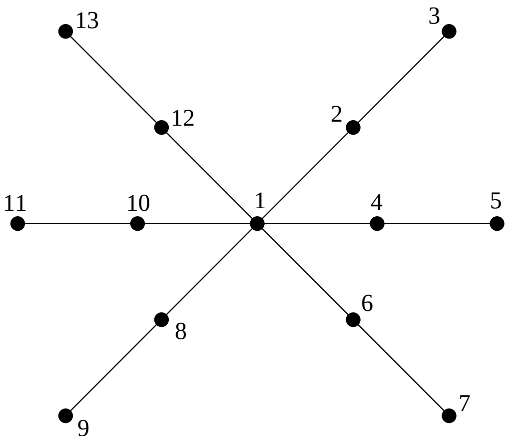

Figure 2: Prime labeling for Double star $K_{1,6,6}$

leads to a result that we can express any two particular positive integers an a sum of three distinct numbers from the set $\{1, 2, \ldots, 2n - 1\}$.

The prime labeling done in above Figure 2 is not unique. In general from the above graph we can say that the structure $K_{1,n,n}$ represents one way of the $2n$ relatively prime pairs of elements of the set $\{1, 2, \ldots, 2n + 1\}$.

Our aim in this paper is to construct a graph that is simple and and has maximum number of edges satisfying the condition of prime labeling. This leads to an interesting result connecting to number theory. Further we investigate the prime labeling for the maximal planar graph class Pl_n.

2 Graph with n Vertices and Maximal Edges Admitting Vertex Prime Labeling

Consider the set $S = \{1, 2, \ldots, 9\}$, containing all natural numbers less than or equal to $n = 9$.

The relatively prime pairs of the set S is

$$\begin{aligned}S' = \{&(1,2),(1,3),(1,4),(1,5),(1.6),(1,7),(1,8),(1,9);\\&(2,3),(2,5),(2,7),(2,9);(3,4),(3,5),(3,7),(3,8);(4,5),(4,7),(4,9);\\&(5,6),(5,7),(5,8),(5,9);(6,7);(7,8),(7,9);(8,9)\}.\end{aligned}$$

$|S| = 9$ and
$|S'| = 8 + 4 + 4 + 3 + 4 + 1 + 2 + 1 = 27.$
Number of relatively prime pair of elements $= 27$.
$\phi(1) = 1$; $\phi(2) = 1$; $\phi(3) = 2$; $\phi(4) = 2$; $\phi(5) = 4$; $\phi(6) = 2$; $\phi(7)$; $\phi(8) = 4$; $\phi(9) = 6$

Number of relatively prime pairs of $|S_n| = \sum_{k=2}^{n} \phi(k)$

Algorithm 2.1
Input: the number of vertices n
Output: the number of the relatively prime pairs "i" and non relative prime pairs "j".
begin
read n
$i = 0$; $j = 0$;
for $k = 1$ to $n - 1$
for $l = k + 1$ to n
if $gcd(k, l) = 1$ then
$i = i + 1$;
else
$j = j + 1$;
print i, j;
end

As per the algorithm $i = 17$ and $j = 4$. We can draw 17 edges for the graph with 7 vertices such that it admits vertex prime labeling. $j = 4$ represents the number of non relatively prime pairs. We construct a upper triangular matrix P_{ij} with entry 1 if i is relatively prime to j otherwise 0. Sum of column wise sum is represent by $p_{1j} + p_{2j} + \cdots + p_{ij}$.
$|S'_n| = \sum_{k=2}^{n} \phi(k)$

	1	2	3	4	5	6	7
1	$\ddots$	1	1	1	1	1	1
2		$\ddots$	1	0	1	0	1
3			$\ddots$	1	1	0	1
4				$\ddots$	1	0	1
5					$\ddots$	1	1
6						$\ddots$	1
7							$\ddots$
$\sum p_{ij} =$	0	1	2	2	4	2	6

$$\sum p_{ij} = 0 + 1 + 2 + 2 + 4 + 2 + 6 = 17$$

$$\sum p_{ij} = 17$$

$$\sum_{k=2}^{7} \phi(k) = 1 + 2 + 2 + 4 + 2 + 6 = 17$$
$$\sum_{i=1}^{n} p_{1j} + p + 2j + \cdots + p_{ij} = \sum p_{ij}$$
$$\sum p_{i1} = 0; \sum p_{i2} = 1; \sum p_{i3} = 2; \sum p_{i4} = 2; \sum p_{i5} = 4; \sum p_{i7} = 6$$
$sump_{i1} = 0$ always.
$$\sum_{i=1}^{n} p_{ik} = \phi(k) \text{ for all } k > 1.$$

The problem can be interpreted as vertex prime labeling in graphs. Given a graph $G(V, E)$ with n vertices is a simple graph with no loops and parallel edges. The vertices $\{v_1, v_2, \ldots, v_n\}$ are labeled with the integers $\{1, 2, \ldots, n\}$ in such a way that for each edge $v_iv_j \in E$, $f(v_i)$ and $f(v_j)$ are relatively prime. i.e., $f : V \to \{1, 2, \ldots, |V|\}$ is a bijection and if $gcd(f(v_i), f(v_j)) = 1$ there exist an edge $xy \in E$.

The interesting question is given a graph G, Is it possible to do prime labeling. Entriger conjecture that every tree has prime labeling. It is not possible to do prime labeling for any given graph. What will be the maximum number of edges that can be drawn to do vertex prime labeling for a given n vertices in a graph i.e., How many relatively prime pairs are there in the set $S_n = \{1, 2, \ldots, n\}$. This can be calculated by the algorithm.

Theorem 2.1. *A maximal number of edges in a simple vertex prime labeling graph with n vertices is* $\sum_{k=2}^{n} \phi(k)$.

Proof. Consider a graph $G(V, E)$ with n vertices is a simple graph with no loops and parallel edges. The vertices $\{v_1, v_2, \ldots, v_n\}$ are labeled with the integers $\{1, 2, \ldots, n\}$ in such a way that for each edge $v_iv_j \in E$, $f(v_i)$ and $f(v_j)$ are relatively prime. i.e., $f : V \to \{1, 2, \ldots, |V|\}$ is a bijection and if $gcd(f(v_i), f(v_j)) = 1$ there exist an edge $xy \in E$.

Now we have to prove that the number of edges in G is equal to $\sum_{k=2}^{n} \phi(k)$. We construct a upper triangular matrix P_{ij} with entry 1 if i is relatively prime to j otherwise 0. The total number of 1's in the matrix is equal to the total number of edges. Hence the total number of edges in the graph ie equal to $\sum_{k=2}^{n} \phi(k)$.

3 Planar Graph with Maximal Edges and Prime Labeling

A graph is planar if it has an embedding on the plane. K_1, K_2, K_3 and K_4 are planar. But K_n for $n \geq 5$ is non-planar. We now introduce a new construction of planar graphs from K_n ($n \geq 5$) [1]. Let $G = (V, E)$ be a graph with $n = |V|$ and $m = |E|$. Let us consider the algebraic definition of a graph G to be a finite nonempty set V together with a symmetric irreflexive binary relation A on V. $E = \{(u, v)/u, v \in V; uAv\}$. The elements of the set E are called the edges of the graph. We deal with a complete graphs K_n which has n vertices and with all possible edges $m = n(n-1)/2$. For example $K_5 = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$. On removal of the edge $(3, 5)$ we get a planar graph.

Similarly the following set of edges can be removed from K_6, K_7 and K_8 respectively to make them planar, $\{(3, 5), (3, 6), (4, 6)\}$, $\{(3, 5), (3, 6), (3, 7), (4, 6), (4, 7), (5, 7)\}$ and $\{(3, 5), (3, 6), (3, 7), (3, 8), (4, 6), (4, 7), (4, 8), (5, 7), (5, 8), (6, 8)\}$. In general we exclude the edges $\{(k, 1) : 3 \leq k \leq n-2; k+2 \leq l \leq n\}$ from K_n to obtain a new class of graph $Pl_n : n \geq 5$ which are planar. □

Definition 3.1. *The graph* $Pl_n = (V, E)$*, where* $V = \{1, 2, \ldots, n\}$ *and* $E = E(K_n)\backslash\{(k, l) : 3 \leq k \leq n-2$ *and* $k+2 \leq l \leq n\}$ *is a planar graph having maximum number of edges, with n vertices.*

The planar graph Pl_n *having maximum number of edges with n vertices is obtained by removal of* $[(n-4)(n-3)]/2$ *edges from* K_n*. The number of edges in* $Pl_n : n \geq 5$ *is* $3(n-2)$*.*

It is interesting to check whether prime labeling is possible for Pl_n *class. Investigations show that* Pl_n *does not admits prime labeling for n even.*

Example 3.1. Pl_{15} *admits vertex prime labeling*

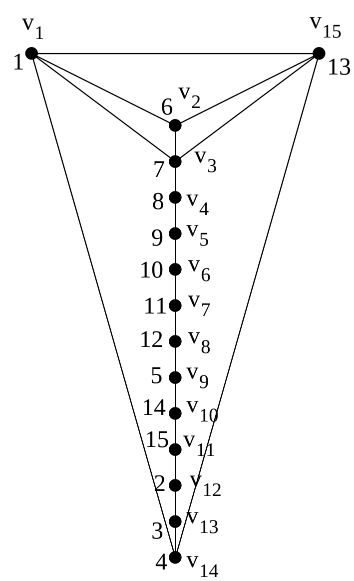

Figure 3:

Algorithm 3.3 (Algorithm for Vertex prime labeling for Pl_n is odd)
Input: n number of vertices of Pl_n,
Output: vertex prime labeling for Pl_n
begin
Step 1: $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_n\} \cup \{v_1v_i, v_nv_i : i = 2 \text{ to } n-1\} \cup \{v_jv_{j+1} : j = 2 \text{ to } n-2\}$
Step 2: the maximal prime number $p < n$ and the minimum
Prime number $p^1 < p$ such that $gcd(p^1, p-1) = gcd(p^1, p+1) = 1$.
Step 3: $f(v_1) = 1$; $f(v_n) = p$;
Step 4: $f(v_i) = p^1 + i - 1$; $2 \leq i \leq p - p^1 - 1$
Step 5: $f(v_{p-p^1}) = p - 1$;
Step 6: $f(v_{p-p^1+1}) = p^1$;
Step 7: $f(v_{p-p^1+2}) = p + 1$; $f(v_{p-p^1+3}) = p + 2, \ldots, f(v_{n-p^1+1}) = n$;
Step 8: $f(v_{n-i}) = p^1 - i$; $1 \leq i \leq p^1 - 2$
end

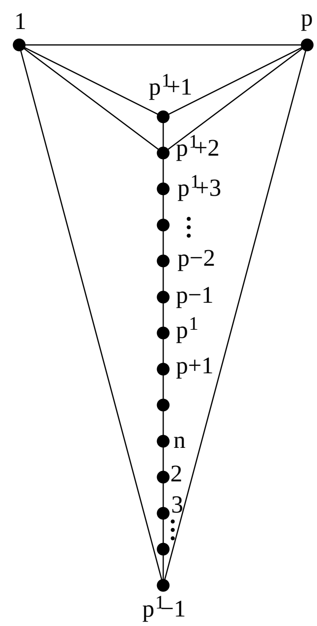

Figure 4:

Theorem 3.1. *The class of Planar graphs Pl_n : n odd admits vertex prime labeling.*

Proof. By the construction of the class Pl_n, n is always greater than or equal to 5.
Case (i) If n **is a prime** then the proof is trivial. We assign $f(v_i) = i$. Since degree of v_1 and v_n is $n - 1$ (adjacent to all vertices) 1 and n are relatively

prime to all numbers $\{2, 3, \ldots, n-1\}$.
Case(ii) If **n is non prime odd number**, then by the algorithm 3.3 we assign the labels for the vertices. By our choice of p^1, it is relatively prime to $p-1$ and $p+1$. Since consecutive numbers are relatively prime and $gcd(2, n) = 1$. The assignment of labels admits prime labeling for the class $Pl_n : n$ odd. □

4 Conclusion

Our results also show some interesting interplay between graph labeling and number theory. The total number of vertices in Pl_n class is $3(n-2)$ and maximum possible edges in a prime labeling graph with n vertices is $\sum_{k=2}^{n} \phi(k)$. It can be verified that $3(n-2) < \sum_{k=2}^{n} \phi(k) < n(n-1)/2$. But only for n, odd the Pl_n class admits prime labeling. This concludes that prime labeling depends only on the structure of the graph and not on the number of edges. The interpretence of labeling may lead to new number theoretical results in vertex prime labeling and bimagic labeling. Further different types of graph labeling can lead to properties of numbers.

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