On Dense Left Cancellative Languages

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Abstract

This paper is dedicated to study dense left cancellative languages. A characterization and some important properties of this class of languages are given by using some class of semi-singular words.

Keywords: Dense left cancellative language, Semi-singular word, Prefix code

1. Introduction

Let $S$ be any semigroup and let

$$D(S) = \{ x \in S | xy = xz \text{ implies } y = z \text{ for all } y, z \in S \}.$$ 

Then $D(S)$ is a subsemigroup of $S$, it is not empty. We call $D(S)$ the left cancellative subsemigroup of $S$. Any element of $D(S)$ is called a left cancellative element of $S$, then $S$ is a left cancellative semigroup. Similarly, we can define right cancellative subsemigroup and right cancellative element of $S$.

Left cancellative elements of the monoid of languages $M = 2^{A^*} \cup \{ \{1\} \}$ are called left cancellative languages. This class of languages has been characterized in [2]. It has been shown that every left singular language is a left cancellative element of $M$. It is clear that $D(M)$ is a monoid. $D(M)$ is a even sp-submonoid of $M([3])$ but not free.

Let $S$ be a semigroup. A subset $X$ of $S$ is called dense in $S$ if it meets all ideals of $S$. A dense subset in a tree monoid $A^*$ over an alphabet $A$ is called a dense language over $A$. The main task of this paper is to study the class
of dense left cancellative languages. We give a complete characterization of this class of languages by using some class of semi-singular words. Items not defined in this paper can be found in the books [4] and [5].

2. Main Results

Let $X \subseteq A^+$. For $v \in \mathbb{Z}^x$ and $x \in A^*$, we call $vx$ $X$-semi-singular if and only if whenever $vxr = yz$ for some $y \in X$, and $r, z \in A^*$, then $y = v$. Let

$$S_X = \{x \in A^+ \mid vx \text{ is } X\text{-semi-singular for some } v \in \mathbb{Z}^x\}.$$  

Let $G_X = S_X / S_XA^+$ be a prefix root of $S_X$. It is immediate that $S_XA^* \subseteq S_X$ and $S_X = G_XA^*$. Let

$$L_X = \{x \in A^+ \mid \text{there exists } p \in X \text{ such that } px \notin XA^+_x \}.$$  

We call $x \in A^+$ $X$-inf-singular if the following conditions hold:

(i) $xA^* \subseteq L_X$.

(ii) $g \neq xm$ and $x \neq gm$ for any $g \in G_X$ (or equivalently $g \in S_X$) and $m \in A^*$.

We denote $I_X = \{x \in A^+ \mid x \text{ is } X\text{-inf-singular} \}$ and $H_X = I_X / I_XA^+$. It is easy to see that $I_XA^+ \subseteq I_X$ and $I_X = H_XA^*$.

**Proposition 1** [1] Let $X \subseteq M$. Then $X \notin D(M)$ if and only if $XA^+ = AA^+_x$ for some $x \in A^+$, where $A^+_x = A^+ / \{x\}$.

**Corollary 2** $X \in D(M)$ if and only if $L_X = A^+$.

**Proposition 3** [1] Let $X \subseteq A^+$, and let $u \in A^+$. Then $u \in S_X$ if and only if $pu \notin XA^+_u$ and $puA^* \cap X = \emptyset$ for some $p \in X$ (or $Z_X$).

**Corollary 4** Let $X \subseteq A^+$, then $S_X \subseteq L_X$.

**Proposition 5** [2] Let $X \subseteq A^+$. Then the following statements are true:

1. $S_X \cap I_X = G_X \cap H_X = \emptyset$.
2. $G_X \cup H_X$ is a prefix code.
3. $X \in D(M)$ if and only if $G_X \cup H_X$ is a maximal prefix code.
Now, we are ready to define $S(M)$ and $I(M)$. Let $X \subseteq A^+$, $X$ is said to be a semi-singular language (resp. inf-singular languages) if $G_X$ (resp. $H_X$) is a maximal prefix code. The class of all semi-singular languages (resp. inf-singular languages) will be denoted by $S(M)$ (resp. $I(M)$). By Proposition 5, $S(M)$ and $I(M)$ are both contained in $D(M)$. Moreover, if $(x) \in S(M)$ (resp. $x \in I(M)$), then $I_X = H_X = \emptyset$ (resp. $S_X = D_X = \emptyset$). It is clear that if $l(X) \neq \emptyset$ then $S_X = A^+$ and $G_X = A$. Hence $S(M)$ contains all left singular languages. It is shown that $I_X = \emptyset$ if $X$ is finite. Hence all finite languages are also contained in $S(M)$, and it has been proved that $S(M)$ is a subsemigroup of $D(M)$ in [4].

Let $X \subseteq A^+$. A word $x \in A^+$ is said to be $X$-thin-singular if and only if $px \not\in XA_X^+$ and $A^*xA^* \cap X = \emptyset$ for some $p \in X$ (or equivalently $p \in Z_X$). Let $S'_X$ be the set of all $X$-thin-singular words and $G'_X$ be the prefix root of $S'_X$. We call $x \in A^+$ $X$-dense-singular if the following conditions hold:

(i) $xA^* \subseteq L_X$;
(ii) $gx \neq xm$ and $x \neq gm$ for any $g \in G'_X$ (or equivalently $g \in S'_X$) and $m \in A^*$.

Let $I'_X$ be the set of all $X$-dense-singular word and $H'_X$ be the prefix root of $I'_X$.

**Proposition 6** Let $X \subseteq A^+$. Then

1. $S'_X \subseteq S_X$ and $I_X \subseteq I'_X$.
2. $S'_XA^* \subseteq S_XA^*$. Hence $S'_X = G'_XA^*$.
3. $I'_XA^* \subseteq I'_X$. Hence $I'_X = H'_XA^*$.

**Proof.** (1) By the definition of $S'_X$ and $I'_X$, this result is obvious.

(2) Let $x \in S'_X$ and $w \in A^*$. Then $xw \in S_XA^* \subseteq A^*$ and $A^*xA^* \cap X = \emptyset$, that is $xw \in S'_X$, hence $S'_X \subseteq S'_X$.

(3) Let $x \in I'_X$ and $w \in A^*$. Since $x \subseteq L_X$. We get $xwA^* \subseteq xA^* \subseteq L_X$. Let $g \in G'_X$ and $m \in A^*$, clearly, $g \neq xwm$. If $xw = gm$, then either $g$ is a prefix of $x$ or $x$ is a prefix of $g$. Which is contradict to $x \in I'_X$. Hence $xw \neq gm$ and $xw \in I'_X$.

**Lemma 7** [4] Let $X$ and $Y$ be two prefix codes. If $XA^* \cap YA^* = \emptyset$, then $X \cup Y$ is a prefix code.

**Theorem 8** Let $X \subseteq A^+$, $T \subseteq L_X$ be any right ideal of $A^*$ and $R = R(T)$ be a language such that $x \in R$ if and only if it satisfies the following two conditions:
(i) \( xA^* \subseteq L_X \);
(ii) \( g \neq xm \) and \( x \neq gm \) for any \( g \in Z_r \) (or equivalently \( g \in T \)) and \( m \in A^* \).

Then

(1) \( T \cap R = Z_r \cap Z_R = \emptyset \).
(2) \( Z_r \cup Z_R \) is a prefix code.
(3) \( X \in D(M) \) if and only if \( Z_r \cup Z_R \) is a maximal prefix code.

**Proof.** (1) By the definition of \( T \) and \( R \).

(2) Since \( T \) is a right ideal of \( A^* \), obviously, we can obtain \( T = Z_r A^* \). Similar to Item (3) of Proposition 6, we can prove that \( R \) is also a right ideal of \( A^* \). Hence \( R = Z_R A^* \). Now, by Item (1), we can get \( Z_r A^* \cap Z_R A^* = T \cap R = \emptyset \), hence \( Z_r \cup Z_R \) is a prefix code by lemma 7.

(3) Since \( X \in D(M) \), we have \( L_X = A^+ \) by corollary 4. Then \( x \in R \) if and only if \( g \neq xm \) and \( x \neq gm \) for any \( g \in Z_r \) and \( m \in A^* \). Let \( x \in A^+ / (Z_r \cup Z_R) \), then there exists \( g \in Z_r \cup Z_R \) such that either \( g \) is a prefix of \( x \) or \( x \) is a prefix of \( g \). It implies that \( Z_r \cup Z_R \) is not prefix code. Hence \( Z_r \cup Z_R \) is a maximal prefix code.

Suppose to the contrary that \( X \notin D(M) \). Then by Proposition 6, we can get \( XA^+ = XA_X^+ \) for some \( x \in A^+ \). Since \( Z_r \cup Z_R \) is a maximal prefix code, we have \( g = xm \) or \( x = gm \) for some \( g \in Z_r \cup Z_R \) and \( m \in A^* \).

(a) If \( g = xm \). Since \( XA^+ = XA_X^+ \), we have \( XA^+ = XA_g^+ \). Hence \( g \notin L_X \), by the definition of \( L_X \). Since \( Z_r \subseteq T \subseteq L_X \) and \( Z_r \subseteq R \subseteq L_X \), we have \( g \notin Z_r \cup Z_R \), this is a contradiction.

(b) If \( x = gm \). Then \( x \in T \subseteq L_X \) which is contradicts to \( XA^+ = XA_X^+ \).

Hence \( X \in D(M) \).

Notice that the above theorem is more general than Proposition 6. Furthermore, we have following corollary.

**Corollary 9** Let \( X \subseteq A^+ \). Then the following are true:

(1) \( S'_X \cap I'_X = G'_X \cap H'_X = \emptyset \).
(2) \( G'_X \cup H'_X \) is a prefix code.
(3) \( X \in D(M) \) if and only if \( G'_X \cup H'_X \) is a maximal prefix code.

**Proof.** Since \( S'_X \subseteq L_X \) and by Item (2) of Proposition 6, \( S'_X \) is a right ideal of \( A^* \). If we let \( T = S'_X \) and \( R = R(T) = I'_X \), then \( T \) and \( R \) satisfy the
condition of Theorem 8. Thus all items in the corollary can be derived from this theorem.

Now, we can define $S'(M)$ and $I'(M)$. Let $X \subseteq A^*$, $X$ is said to be a thin-singular language (resp. dense-singular language) if $G'_X$ (resp. $H'_X$) is a maximal prefix code. The class of all thin-singular languages (resp. dense-singular languages) will be denoted by $S'(M)$ (resp. $I'(M)$). By the above corollary $S$, $S'(M)$ and $I'(M)$ are both contained in $D(M)$. Moreover, if $x \in D(M)$, (resp. $X \in I'(M)$), then $I'_X = H'_X = \emptyset$ (resp. $S'_X = G'_X = \emptyset$). It is clear that $S'(M) \subseteq S(M)$ and $I(M) \subseteq I'(M)$.

**Theorem 10** Let $X \in D(M)$. Then
(1) $X$ is thin if and only if $X$ is thin-singular.
(2) $X$ is dense if and only if $X$ is dense-singular.

**Proof.** (1) if $X \in S'(M)$, then $S'_X \neq \emptyset$. Let $x \in S'_X$. Then $A^*xA^* \cap X = \emptyset$. Hence $X$ is thin.

Conversely, if $X$ is thin, then there exists $w \in A^+$ such that $A^*xA^* \cap X = \emptyset$. Hence $A^*xA^* \cap X = \emptyset$ for any $u \in A^*wA^*$. Since $X \in D(M)$, we have $pu \notin XA^+$ for some $p \in X$. Thus $u \in S'_X$ by the definition of $S'_X$. It suggests that $A^*wA^* \subseteq S'_X$. Now, for any $x \in A^+$, $xw \in A^*wA^* \subseteq S'_X$. It follows that $x \notin I'_A$. Then $H'_X = I'_X = \emptyset$. Since $X \in D(M)$, we have, by Corollary 9, $G'_X$ is a maximal prefix code and $X \in S'(M)$.

(2) if $X \in I'(M)$, then $X \notin S'(M)$. Hence $X$ is dense by Item(1).

Conversely, if $X$ is dense, then for any $x \in A^+$, $A^*wA^* \cap X \neq \emptyset$. Hence $G'_x = S'_X = \emptyset$ and $H'_X$ is a maximal prefix code by Corollary 9, that is $X \in I'(M)$.

**References**


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