Endomorphism Rings of Small Pseudo Projective Modules

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Abstract

In this paper I have tried to find some of the results on endomorphism rings of small pseudo projective modules.

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1 Introduction

Throughout this paper the basic ring $R$ is supposed to be ring with unity and all modules are unitary left $R$-modules.

Let $M$ be an $R$-module, a submodule $K$ of $M$ is said to be small in $M$ if $K + L = M \Rightarrow L = M$ for any submodule $L \subseteq M$. An $R$-module $M$ is said to be hollow if all proper submodules of $M$ are small in $M$. An $R$-module $M$ is said to be small quasi projective if for any module $A$, with small epimorphism $g : M \to A$ and homomorphism $f : M \to A$ there exists an $h \in \text{End}(M)$ such that $f = goh$. An $R$-module $M$ is said to be small pseudo projective if for any module $A$, with small epimorphism $g : M \to A$ and epimorphism $f : M \to A$ there exists an $h \in \text{End}(M)$ such that $f = goh$. A ring $R$ is called regular
(in the sense of Von-Neumann) if for each \( r \in R \) there exists \( x \in R \) such that \( r = rxx \). The jacobson radical \( J(M) \), of a module \( M \), is the intersection of all maximal submodules of \( M \). An \( R \)-module \( M \) is called local if it has a unique maximal submodule which contains every proper submodules of \( M \). The socle of an \( R \) module \( M \) denoted by \( Soc(M) \) is defined as intersection of essential submodules of \( M \). A \( \mathbb{R} \)-module \( M \) is called local if it has a unique maximal submodule which contains every proper submodules of \( M \).

The socle of an \( \mathbb{R} \) module \( M \) denoted by \( Soc(M) \) is defined as intersection of essential submodules of \( M \). Two module epimorphisms \( f, g : P \to M \) are right equivalent if \( f = goh \) for some automorphism \( h \) of \( P \). An \( \mathbb{R} \) module \( M \) is called \( \pi \)-projective if for all submodules \( U \) and \( V \) of \( M \) with \( U + V = M \), there exists \( f \in S \) with \( Imf \subseteq U \) and \( Im(1 - f) \subseteq V \). A submodule \( N \) of an \( \mathbb{R} \)-module \( M \) is said to be small pseudo stable if for any epimorphism \( f : M \to A \) and any small epimorphism \( g : M \to A \) with \( N \subseteq Kerg \cap Kerf \), there exists \( h \in End(M) \) such that \( f = goh \) then, \( h(N) \subseteq N \). A module \( M \) is called a duo module if every submodule of \( M \) is fully invariant.

2 Main Results

**Proposition 1.** Let \( M \) be any small pseudo projective hollow module. Then every epimorphism in \( End(M) \) is an automorphism.

**Proof:** Let \( g : M \to M \) be any epimorphism then we have \( Kerg \neq M \). So, \( Kerg \) is a proper submodule of \( M \). As \( M \) is hollow, \( g \) is a small epimorphism, by small pseudo projectivity of \( M \), \( I_M \) can be lifted to a homomorphism \( h : M \to M \) such that \( goh = I_M \).

\[ \Rightarrow h \text{ is one-one.} \]

Let \( m \in M \) then as \( g \) is onto there exists an element \( n \in M \) such that \( m = g(n) \Rightarrow g(n - h(m)) = 0 \Rightarrow n - h(m) \in Kerg \Rightarrow n \in Kerg + h(m) \Rightarrow M \subseteq Kerg + Imh. \) So, we have \( M = Kerg + Imh \Rightarrow M = Imh \), since \( M \) is hollow. Thus \( h \) is onto and so \( h \) is an automorphism \( \Rightarrow h^{-1} = g \) is an automorphism.

**Proposition 2.** (a) If \( S \) is the endomorphism ring of a small quasi projective hollow module \( M \) then \( S \) is local.

(b) If \( S \) is the endomorphism ring of a small pseudo projective hollow module \( M \) then \( S \) is local.

**Proof:** Follows from [1, Theorem 1.14]

**Proposition 3.** Let \( M \) be any pseudo projective module and \( End(M) \) denotes the endomorphism ring of \( M \). Then if \( \alpha(M) \subseteq^\oplus M \) for every \( \alpha \in End(M) \) then \( ker\alpha \subseteq^\oplus M \).

**Proof:** Follows from [7, Proposition 8].
Proposition 4. Let $M$ be any pseudo projective module and $\text{End}(M)$ denotes the endomorphism ring of $M$. Then if $\alpha(M) \subseteq M$ for every $\alpha \in \text{End}(M)$ then $\text{End}(M)$ is regular.

Proof: Follows from [7, Proposition 10].

Corollary 4.1: Endomorphism ring of a completely reducible pseudo projective module is regular.

Proposition 5. Let $M$ be a small pseudo projective hollow module $S$ denotes the endomorphism ring of $M$, $J(S)$ denotes the jacobson radical of $S$ then
(a) $J(S) = \{ \alpha \in S | \text{Im} \alpha \text{ is small in } M \}$
(b) $J(S) \subseteq \text{Hom}(M, J(M))$
(c) $S/J(S)$ is von-neumann regular ring.

Proof: Follows from [1, Theorem 1.15]

Proposition 6. Let $M$ be a small pseudo projective hollow module and $K$ be any small submodule of $M$ then for any automorphism $g \in \text{Aut}(M/K)$ there exists an automorphism $h \in \text{Aut}(M)$ such that $g(m + K) = h(m) + K \ \forall m \in M$.

Proof: Let $K$ be any small submodule of $M$ and $\nu : M \rightarrow M/K$ be any natural map, and $g : M/K \rightarrow M/K$ be any automorphism in $\text{Aut}(M/K)$. Then by small pseudo projectivity of $M \exists h \in \text{End}(M)$ such that $g \nu = \nu h$ i.e. $g \nu(m) = \nu h(m) \forall m \in M \Rightarrow g(m + K) = h(m) + K \ \forall m \in M$. Then by [5, Proposition 4] $h$ is an epi-endomorphism. By Proposition 1 we get $h$ is an automorphism.

Proposition 7. Let $M$ be a small pseudo projective hollow module then for any $\alpha \in \text{End}(M)$ and any small submodule $K$ of $M$ with $\alpha(M) + K = M$ and $\alpha^{-1}(K) = K$ there exists $\beta \in \text{End}(M)$ such that $\beta(M) \subseteq K$ and $\alpha + \beta \in \text{Aut}(M)$.

Proof: Suppose $\alpha \in \text{End}(M)$ and $K$ is any small submodule of $M$ satisfying $\alpha(M) + K = M$ and $\alpha^{-1}(K) = K$. Let $f : M \rightarrow M/K$ be the natural map. Now we have $\text{Ker}(f \alpha) = \alpha^{-1}(\text{Ker} f) = \alpha^{-1}(K) = K = \text{Ker} f$. Thus, $\text{Ker}(f \alpha) = \text{Ker} f$. Now, $\alpha(M) + K = M \Rightarrow \alpha(M) = M \Rightarrow \alpha$ is onto and therefore $f \alpha$ is onto. So by [2, Theorem 3.6] $\exists$ an automorphism $g \in \text{End}(M/K) \ni \text{gof} = f \alpha$. So by assumption there exists $h \in \text{Aut}(M)$ such that $g(m + K) = h(m) + K \Rightarrow g(M/K) = foh(M) \Rightarrow \text{gof} (M) = foh(M) \Rightarrow \text{gof} = foh \Rightarrow f \alpha = foh \Rightarrow f(h - \alpha) = 0$. Let $\beta = h - \alpha$. We have $\beta(M) \subseteq K$. Also $\alpha + \beta = h$ is an automorphism in $\text{Aut}(M)$. 
Proposition 8. Let $M$ be a small pseudo projective hollow module then any pair of small epimorphisms from $M$ to any module $N$ are right equivalent if for given any $\alpha \in \text{End}(M)$ and any small submodule $K$ of $M$ there exists $\beta \in \text{End}(M)$ such that $\beta(M) \subseteq K$ and $\alpha + \beta \in \text{Aut}(M)$.

Proof: Suppose $f, g : M \to N$ are small epimorphism. By small pseudo projective of $M$ there exists $\alpha \in \text{End}(M)$ such that $f = go\alpha$. Since $f$ is epimorphism we have $\alpha(M) + \text{Ker}(g) = M$ then by assumption there exists $\beta \in \text{End}(M)$ such that $\alpha + \beta \in \text{Aut}(M)$ and $\beta(M) \subseteq K$. So $g(\alpha + \beta) = go\alpha + go\beta = go\alpha = f$. So, $f$ and $g$ are right equivalent.

Proposition 9. Let $M$ be a duo and small pseudo projective module. Let $S$ denotes the endomorphism ring of $M$ and $T = \{\alpha \in S|\text{Im } \alpha \text{ is small in } M\}$. Then for every $f \in T$, $\text{Im } f$ is a small pseudo stable submodule of $M$.

Proof: Let $f \in T$ then $\text{Im } f$ is a small submodule of $M$. Let $g : M/\text{Im } f \to A$ be a small epimorphism, $\psi : M/\text{Im } f \to A$ be an epimorphism and $\nu : M \to M/\text{Im } f$ be the natural map. Then $\text{Ker } \nu = \text{Im } f$ is a small submodule of $M \Rightarrow \nu$ is a small epimorphism. Now, $\text{Im } f \subseteq \text{Ker}(g\nu) \cap \text{Ker}(\psi \nu)$, since $\text{Ker } \nu = \text{Im } f \Rightarrow \nu(\text{Im } f) = 0 \Rightarrow g(\nu(\text{Im } f)) = 0 \Rightarrow \text{Im } f \subseteq \text{Ker}(g\nu)$. Similarly $\text{Im } f \subseteq \text{Ker}(\psi \nu))$. By small pseudo projectivity of $M$ there exists $h \in \text{End}(M)$ such that $\psi \nu = g\nu h$. We have, $h(\text{Im } f) \subseteq \text{Im } f$, since $M$ is duo and $\text{Im } f \subseteq M$. So, $\text{Im } f$ is a small pseudo stable submodule of $M$.

Proposition 10. Let $M$ be a duo and small pseudo projective hollow module. Let $S$ denotes the endomorphism ring of $M$ and $J(S)$ denotes the jacobson radical of $M$. Then for every $f \in J(S)$, $\text{Im } f$ is a small pseudo stable submodule of $M$.

Proof: By Proposition 5(a), we have $T = J(S)$. Rest of the proof follows from Proposition 9.

Proposition 11. Let $M$ be a small pseudo projective module if $S$ is local and $M$ is $\pi$-projective then $M$ is hollow.

Proof: Let $U$ and $V$ be submodules of $M$ such that $U + V = M$. As $M$ is $\pi$-projective there exists $f \in S$ such that $\text{Im } f \subseteq U$ and $\text{Im }(1 - f) \subseteq V$. Now $S$ is local so, $f \in S \Rightarrow$ either $f$ or $(1 - f)$ is invertible. Now $f$ is invertible $\Rightarrow \exists$ $g \in S \ni fog = I_M \Rightarrow f$ is onto and so $\text{Im } f = M \Rightarrow U = M$. Thus $V$ is small. Similarly we can show that when $(1 - f)$ is invertible then $V = M$ $\Rightarrow U$ is small, and therefore $M$ is hollow.

Proposition 12. Let $M$ be a small pseudo projective $D2$ module. Then $M$ is $S.F.$

Proof: Follows from [5, Proposition 3]
Proposition 13. Let $M$ be a small quasi projective duo module. If $S = \text{End}(M)$, is local, then $M$ is not supplemented.

Proof: Suppose that $M$ is supplemented and $A$ is any submodule of $M$. Let $B$ be supplement of $A$ in $M$ then we have $M = A + B$ and $A \cap B$ is small in $M$. Let $0 \neq s(M) = A$ and $0 \neq t(M) = B$, $s,t \in S$. Define the map $f : M = (s + t)(M) \to M/(A \cap B)$ such that $f(s + t)(m) = s(m) + (A \cap B)$. For any $m, m' \in M$, $(s + t)(m) = (s + t)(m')$ implies that $s(m - m') = t(m' - m) \in A \cap B$. So $s(m) + (A \cap B) = s(m') + (A \cap B)$. Thus $f$ is well defined and $f$ is also an $R$-homomorphism. Let $\nu : M \to M/(A \cap B)$ is the natural map. By small quasi projectivity of $M$, there exist $g \in S$ such that $\nu og = f$. We have $\nu og(s + t)(m) = f(s + t)(m) = s(m) + (A \cap B)$. Then, $g(s + t)(m) + (A \cap B) = s(m) + (A \cap B) \Rightarrow ((1 - g)os - got)(M) \subseteq (A \cap B)$. Since $S$ is local, $g$ or $(1 - g)$ is invertible. If $(1 - g)$ is invertible we have, $(s - (1 - g)^{-1}got)(M) \subseteq (1 - g)^{-1}(A \cap B) \subseteq (A \cap B)$. Now $A \subseteq (s - (1 - g)^{-1}got)(M) \subseteq (A \cap B)$. Then $A \subseteq (A \cap B)$, which is a contradiction. Similarly if $g$ is invertible we have $B \subseteq (g^{-1}os - t) \subseteq g^{-1}(A \cap B) \subseteq (A \cap B)$. Then $B \subseteq (A \cap B)$, that is also a contradiction. Hence $M$ is not supplemented.

Corollary 13.1: Let $M$ be a hollow small quasi projective duo module. Then $M$ is not supplemented.

Proof: Follows from Proposition 2(a) and Proposition 13.

References


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