Ultra-Hyperbolic Wave Operator Related to Nonlinear Wave Equation

Wanchak Satsanit

Department of Mathematics
Chiangmai University
Chiangmai, 50200, Thailand
aunphue@live.com

Amnuay Kananthai

Department of Mathematics
Chiangmai University
Chiangmai, 50200, Thailand
malamnka@science.cmu.ac.th

Abstract

In this paper, we study the generalized wave equation of the form
\[
\frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box) u(x, t) = f(x, t, u(x, t))
\]
with the initial conditions
\[
u(x, 0) = f(x), \quad \frac{\partial}{\partial t} u(x, 0) = g(x)
\]
where \(u(x, t) \in \mathbb{R}^n \times [0, \infty)\), \(\mathbb{R}^n\) is the \(n\)-dimensional Euclidean space, \(\Box\) is the ultra-hyperbolic operator defined by
\[
\Box = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right),
\]
p + q = n, \(c\) is a positive constant, \(f\) and \(g\) are continuous and absolutely integrable functions. By \(\epsilon\)-approximation we also obtain the asymptotic solution \(u(x, t) = O(\epsilon^{-n}) \ast f(x, t, u(x, t))\). In particularly, if we put, \(q = 0\), the \(u(x, t)\) reduces to the solution of the \(n\)-dimensional nonlinear wave equation
\[
\frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Delta) u(x, t) = f(x, t, u(x, t))
\]
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1 Introduction

It is well known that for the 1-dimensional wave equation
\[ \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t), \]  
we obtain \( u(x, t) = f(x + ct) + g(x - ct) \) as a solution of the equation where \( f \) and \( g \) are continuous.

Also for the \( n \)-dimensional wave equation
\[ \frac{\partial^2}{\partial t^2} u(x, t) + c^2 \Delta u(x, t) = 0, \]  
with the initial condition
\[ u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = g(x) \]
where \( f \) and \( g \) are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by
\[ \hat{u}(\xi, t) = \hat{f}(\xi) \cos(2\pi|\xi|) t + \hat{g}(\xi) \frac{\sin(2\pi|\xi|) t}{2\pi|\xi|} \]
where \( |\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 \) (see [2]). By using the inverse Fourier transform, we obtain \( u(x, t) \) in the convolution form, that is
\[ u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \]  
where \( \Phi_t \) is an inverse Fourier transform of \( \hat{\Phi}_t(\xi) = \frac{\sin(2\pi|\xi|) t}{2\pi|\xi|} \) and \( \Psi_t \) is an inverse Fourier transform of \( \hat{\Psi}_t(\xi) = \cos(2\pi|\xi|) t = \frac{\partial}{\partial t} \hat{\Phi}(\xi) \).

Next, G. Sritantana, A. Kananthai study the equation
\[ \frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\Delta)^k u(x, t) = 0 \]
see[3], where
\[ \Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \]
We obtain the solution related to the beam equation. Next, W. Satsanit, A. Kananthai study equation
\[ \frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box)^k u(x, t) = 0 \]
see[4], where
\[ \Box^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \]
We obtain the solution related to the beam equation also. In this paper, we study the equation
\[ \frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box) u(x, t) = f(x, t, u(x, t)) \tag{1.4} \]
which is in the form of nonlinear wave equation. We consider (1.4) with the following conditions on \( u \) and \( f \) as follows

(1) \( u(x, t) \) is the space of continuous function on \( \mathbb{R}^n \times (0, \infty) \).

(2) \( f \) satisfies the Lipchitz condition,
\[ |f(x, t, u) - f(x, t, w)| \leq A|u - w| \]
where \( A \) is constant with \( 0 < A < 1 \).

(3) \[ \int_0^\infty \int_{\mathbb{R}^n} |f(x, t, u(x, t))| \, dx \, dt < \infty \] for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( 0 < t < \infty \) and \( u(x, t) \) is continuous function on \( \mathbb{R}^n \times (0, \infty) \).

Under such conditions of \( f \) and \( u \) and for the spectrum of \( E(x, t) \), we obtain the convolution (see [1, p150-153]).
\[ u(x, t) = E(x, t) \ast f(x, t, u(x, t)) \tag{1.5} \]
as a unique solution of (1.4) where \( E(x, t) \) is an elementary solution of (1.4). with \( u(x, 0) = f(x) \) and \( \frac{\partial}{\partial t} u(x, 0) = g(x) \) where \( c \) is a positive constant, \( f \) and \( g \) are continuous functions and absolutely integrable.

Moreover, if we put \( q = 0 \) in (1.4) then (1.5) reduces to the solution of the \( n \)-dimensional nonlinear wave equation.
2 Preliminaries

We shall need the following definitions

**Definition 2.1.** Let $f \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi \cdot x)} f(x) \, dx$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi \cdot x)} \hat{f}(x) \, dx$$

**Definition 2.2.** Let $t > 0$ and $p$ is a real number

- $f(t) = O(t^p)$ as $t \to 0 \iff t^{-p}|f(t)|$ is bounded as $t \to 0$
- and $f(t) = o(t^p)$ as $t \to 0 \iff t^{-p}|f(t)| \to 0$ as $t \to 0$

**Lemma 2.3.** Given the function

$$f(x) = \exp \left[ - \sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2 \right]$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. Then

$$\left| \int_{\mathbb{R}^n} f(x) \, dx \right| \leq \frac{\Omega_p \Omega_q}{2} \cdot \frac{\Gamma(n)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2+q}{2}\right)}$$

where $\Gamma$ denotes the Gamma function. That is $\int_{\mathbb{R}^n} f(x) \, dx$ is bounded.

**Proof.**

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} \exp \left[ - \sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2 \right] \, dx$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \quad x_2 = r\omega_2, \ldots, \quad x_p = r\omega_p$$

$$dx_1 = rd\omega_1, \quad dx_2 = rd\omega_2, \ldots, \quad dx_p = rd\omega_p$$
and

\[ x_{p+1} = s \omega_{p+1}, \quad x_{p+2} = s \omega_{p+2}, \ldots, \quad x_{p+q} = s \omega_{p+q} \]
\[ dx_{p+1} = s d \omega_{p+1}, \quad dx_{p+2} = s d \omega_{p+2}, \ldots, \quad dx_{p+q} = s d \omega_{p+q} \]

where \( \omega_1^2 + \omega_2^2 + \ldots + \omega_p^2 = 1 \) and \( \omega_{p+1}^2 + \omega_{p+2}^2 + \ldots + \omega_{p+q}^2 = 1 \).

Thus

\[ \int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} \exp \left[ -\sqrt{s^2 - r^2} \right] r^{p-1} s^{q-1} d r d s \Omega_p d \Omega_q \]

where \( d x = r^{p-1} s^{q-1} d r d s \Omega_p d \Omega_q \), \( d \Omega_p \) and \( d \Omega_q \) are the elements of surface area on the unit sphere in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively,

\[ |\int_{\mathbb{R}^n} f(x) \, dx| \leq \Omega_p \Omega_q \int_0^\infty \int_0^s \exp \left[ -\sqrt{s^2 - r^2} \right] r^{p-1} s^{q-1} d r d s. \]

By computing directly, we obtain

\[ \int_{\mathbb{R}^n} f(x) \, dx = \Omega_p \Omega_q \int_0^\infty \int_0^s \exp \left[ -\sqrt{s^2 - r^2} \right] r^{p-1} s^{q-1} d r d s \]

where \( \Omega_p = \frac{2 \pi^{p/2}}{\Gamma(p/2)} \) and \( \Omega_q = \frac{2 \pi^{q/2}}{\Gamma(q/2)} \). Thus

\[ |\int_{\mathbb{R}^n} f(x) \, dx| \leq \Omega_p \Omega_q \int_0^\infty \int_0^s \exp \left[ -\sqrt{s^2 - r^2} \right] r^{p-1} s^{q-1} d r d s. \]

Put \( r = s \sin \theta \), \( dr = s \cos \theta d \theta \) and \( 0 \leq \theta \leq \frac{\pi}{2} \),

\[ |\int_{\mathbb{R}^n} f(x) \, dx| \leq \Omega_p \Omega_q \int_0^\infty \int_0^s e^{s^2 - s^2 \sin^2 \theta} (s \sin \theta)^{p-1} s^{q-1} s \cos \theta d \theta d \theta \]

\[ = \Omega_p \Omega_q \int_0^\infty \int_0^s e^{-s \cos \theta} s^{p+q-1} (s \sin \theta)^{p-1} \cos \theta d \theta d \theta. \]

Put \( y = s \cos \theta \), \( ds = \frac{dy}{\cos \theta} \),

\[ |\int_{\mathbb{R}^n} f(x) \, dx| \leq \Omega_p \Omega_q \int_0^{\pi/2} \int_0^\infty e^{-y} \left( s \frac{y}{\cos \theta} \right)^{n-1} (s \sin \theta)^{p-1} \cos \theta d \theta d \theta \]

\[ = \Omega_p \Omega_q \int_0^{\pi/2} \int_0^\infty e^{-y} y^{n-1} (\cos \theta)^{1-n} (s \sin \theta)^{p-1} d y d \theta \]

\[ = \Omega_p \Omega_q \Gamma(n) \int_0^{\pi/2} (\cos \theta)^{1-n} (s \sin \theta)^{p-1} d \theta \]

\[ = \Omega_p \Omega_q \Gamma(n) \frac{\beta \left( \frac{p}{2}, \frac{2-n}{2} \right)}{\frac{2}{2}} \]

\[ |\int_{\mathbb{R}^n} f(x) \, dx| \leq \Omega_p \Omega_q \Gamma(n) \frac{\beta \left( \frac{p}{2}, \frac{2-n}{2} \right)}{\Gamma \left( \frac{2-q}{2} \right)}. \]

That is \( \int_{\mathbb{R}^n} f(x) \, dx \) is bounded.
Lemma 2.4. Given the operator

\[ L = \frac{\partial^2}{\partial t^2} + c^2(\Box) \] (2.3)

Where \( \Box \) is the Ultrahyperbolic operator defined by

\[ \Box = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right), \]

where \( p+q = n \) is the dimension of Euclidean space \( \mathbb{R}^n \), \( (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( c \) is a positive constant. Then we obtain

\[ E(x,t) = O(\epsilon^{-n}). \] (2.4)

Where \( E(x,t) \) is the elementary solution for the operator \( L \) defined by (2.3)

Proof. We have to find function \( E(x,t) \) from the equation

\[ L(E(x,t)) = \delta(x,t) \]

where \( \delta(x,t) \) is Dirac delta function for \( (x,t) \in \mathbb{R}^n \times (0, \infty) \). We can also write

\[ \frac{\partial^2}{\partial t^2}E(x,t) + c^2(\Box)E(x,t) = \delta(x) \cdot \delta(t) \] (2.5)

By taking the Fourier transform defined by (2.1) to both sides of (2.5), we obtain

\[ \frac{\partial^2}{\partial t^2}\hat{E}(\xi,t) + c^2((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2) - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2))\hat{E}(\xi,t) = \delta(t) \]

The solution of (2.5) is \( \hat{E}(\xi,t) = H(t)\phi(\xi,t) \)

Where \( H(t) \) is a heaviside function and \( \phi(\xi,t) \) is a solution of homogeneous equation and satisfied initial conditions. Now, we are solving the solution of homogeneous equation. Given the homogeneous equation

\[ \frac{\partial^2}{\partial t^2}u(x,t) + c^2(\Box)u(x,t) = 0 \] (2.6)

with the initial condition

\[ u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x) \] (2.7)

By applying the Fourier transform defined by (2.1) to (2.6), we obtain

\[ \frac{\partial^2}{\partial t^2}\hat{u}(\xi,t) + c^2(s^2 - r^2)\hat{u}(\xi,t) = 0 \]
where \( r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 \) and \( s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2 \).

\[
\hat{u}(\xi, t) = A(\xi) \cos c\left(\sqrt{s^2 - r^2}\right)t + B(\xi) \sin c\left(\sqrt{s^2 - r^2}\right)t.
\]

By (3.2), \( \hat{u}(\xi, 0) = A(\xi) = \hat{f}(\xi) \)

\[
\frac{\partial \hat{u}(\xi, t)}{\partial t} = -c \left(\sqrt{s^2 - r^2}\right) A(\xi) \sin c\left(\sqrt{s^2 - r^2}\right)t + c \left(\sqrt{s^2 - r^2}\right) B(\xi) \cos c\left(\sqrt{s^2 - r^2}\right)t.
\]

\[
B(\xi) = \frac{\hat{g}(\xi)}{c \left(\sqrt{s^2 - r^2}\right)}
\]

\[
\phi(\xi, t) = \hat{u}(\xi, t) = \hat{f}(\xi) \cos c\left(\sqrt{s^2 - r^2}\right)t + \frac{\hat{g}(\xi)}{c \left(\sqrt{s^2 - r^2}\right)} \sin c\left(\sqrt{s^2 - r^2}\right)t
\]

(2.8)

Thus

\[
E(\xi, t) = H(t)\hat{f}(\xi) \cos c\left(\sqrt{s^2 - r^2}\right)t + \frac{\hat{g}(\xi)}{c \left(\sqrt{s^2 - r^2}\right)} \sin c\left(\sqrt{s^2 - r^2}\right)t
\]

(2.9)

By applying the inverse Fourier transform (2.9), we obtain the solution \( E(x, t) \) in the convolution form of

\[
E(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x)
\]

(2.10)

Where \( \Phi_t(x) \) and \( \Psi_t(x) \) are the inverse transform of

\[
\hat{\Phi}_t(\xi) = \frac{\sin c\left(\sqrt{s^2 - r^2}\right)t}{c \left(\sqrt{s^2 - r^2}\right)} \quad \text{and} \quad \Psi_t(\xi) = \cos c\left(\sqrt{s^2 - r^2}\right)t.
\]

respectively.

Now we need to show the existence of \( \Phi_t(x) \) and \( \Psi_t(x) \).

Let us consider the Fourier transform

\[
\hat{\Phi}_t(x) = \frac{\sin c\left(\sqrt{s^2 - r^2}\right)t}{c \left(\sqrt{s^2 - r^2}\right)} \quad \text{and} \quad \Psi_t(x) = \cos c\left(\sqrt{s^2 - r^2}\right)t.
\]

They are all tempered distributions but they are not \( L_1(\mathbb{R}^n) \) the space of integrable function. So we cannot compute the inverse Fourier transform \( \Phi_t(x) \)
and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of $\epsilon-$approximation.

Let us defined 

$$\widehat{\Phi}_t(\xi) = e^{-\epsilon c(\sqrt{s^2-r^2})} \Phi_t(\xi) = e^{-\epsilon c(\sqrt{s^2-r^2})} \sin c \left( \sqrt{s^2-r^2} t \right) \frac{c \left( \sqrt{s^2-r^2} \right)}{c \left( \sqrt{s^2-r^2} \right)} \quad \text{for} \quad \epsilon > 0.$$  

(2.11)

We see that $\phi'_t(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi}_t(x) \to \phi_t(x)$ uniformly as $\epsilon \to 0$. So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi'_t(x)$. Now 

$$\Phi_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{\Phi}_t(\xi) d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-\epsilon c(\sqrt{s^2-r^2})} \sin c \left( \sqrt{s^2-r^2} t \right) \frac{c \left( \sqrt{s^2-r^2} \right)}{c \left( \sqrt{s^2-r^2} \right)} d\xi$$

$$|\Phi_t(x)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\epsilon c(\sqrt{s^2-r^2})} r^{p-1}s^{q-1} drdsd\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1} drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, where $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$.

By changing to bipolar coordinates. Now, put 

$$\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p$$

and 

$$\xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_p = sw_{p+q}, \ p + q = n$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$,

$$|\Phi_t(x)| \leq \frac{(p^*)_{\Omega_p \Omega_q}}{2\pi^n} \int_{0}^{\infty} \int_{0}^{\infty} s^{p-1} \left( \sqrt{s^2 - r^2} \right) drds, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

put $r = s \sin \theta$, $dr = s \cos \theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$.

$$|\Phi_t(x)| \leq \frac{(p^*)_{\Omega_p \Omega_q}}{2\pi^n} \int_{0}^{\pi/2} \int_{0}^{s \cos \theta} \frac{e^{-\epsilon c(\sqrt{s^2 - s^2 \sin^2 \theta})}}{c \left( \sqrt{s^2 - s^2 \sin^2 \theta} \right)} (s \sin \theta)^{p-1}s^{q-1}s \cos \theta d\theta ds,$$

$$= \frac{(p^*)_{\Omega_p \Omega_q}}{c(2\pi)^n} \int_{0}^{\pi/2} \int_{0}^{s \cos \theta} \frac{e^{-\epsilon c(s \cos \theta)}}{(s \cos \theta)^{p-1}s^{q-1}s(s \sin \theta)^{p-1} \cos \theta d\theta ds.}$$
Put $y = e c (s \cos \theta) = \epsilon c s \cos \theta$, $s = \frac{y}{e c \cos \theta}$, $ds = \frac{dy}{cs} \cos \theta = \frac{dy}{y}$, thus

$$|\Phi'_t(x)| \leq \frac{\Omega_p \Omega_q}{c(2\pi)^n} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y s^{n-1}}}{y/(e c)} (\sin \theta)^{p-1} \cos \theta^{s} \frac{dy d\theta}{y}$$

$$= \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}}{y^2} \left(\frac{y}{e c \cos \theta}\right)^n (\sin \theta)^{p-1} \cos \theta^{s} dy d\theta$$

$$= \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}}{e^{n-1}} (\sin \theta)^{p-1} (\cos \theta)^{1-n} dy d\theta$$

$$= \frac{\Omega_p \Omega_q}{(2\pi)^n} \Gamma(n-1) \int_0^{\pi/2} (\sin \theta)^{p-1} (\cos \theta)^{1-n} d\theta$$

$$= \frac{\Omega_p \Omega_q}{2e^n(2\pi)^n e^{n-1}} \frac{\Gamma(n-1)}{\Gamma(\frac{q}{2})} \frac{\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-q}{2})}$$

Similarly, we defined $\hat{\Psi}'_t(x) = e^{-\epsilon c(\sqrt{s^2-r^2})} \cos c \left(\sqrt{s^2-r^2}\right) t$ and

$$\Psi'_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{\Psi}'_t(\xi) d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-\epsilon c(\sqrt{s^2-r^2})} \cos c \left(\sqrt{s^2-r^2}\right) t d\xi$$

$$|\Psi'_t(x)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\epsilon c(\sqrt{s^2-r^2})} d\xi$$

$$= \frac{1}{(2\pi)^n} \int_0^\infty \int_0^\infty e^{-\epsilon c(\sqrt{s^2-r^2})} r^{p-1} s^{q-1} dr ds,$$

put $r = s \sin \theta$, $dr = s \cos \theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$

$$|\Psi'_t(x)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^{\pi/2} \int_0^\infty e^{-\epsilon c(s \cos \theta)} (s \sin \theta)^{p-1} s^{q-1} s \cos \theta d\theta ds$$

$$= \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^{\pi/2} \int_0^\infty e^{-\epsilon c(s \cos \theta)} s^{p+q-1} (\sin \theta)^{p-1} \cos \theta d\theta ds,$$
put \( y = \epsilon c(s \cos \theta) \), \( ds = \frac{dy}{y} \),

\[
|\Psi_t^\epsilon(x)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}}{y} \left( \frac{y}{\epsilon c \cos \theta} \right)^n (\sin \theta)^{p-1} \cos \theta dyd\theta
\]

\[
= \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}y^{n-1}}{c^n \epsilon^n} (\sin \theta)^{p-1} (\cos \theta)^{1-n} dyd\theta
\]

\[
|\Psi_t^\epsilon(x)| \leq \frac{\Omega_p \Omega_q}{2(2\pi)^n c^n \epsilon^n} \frac{\Gamma(n) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-n}{2}\right)}
\]

Set

\[
E^\epsilon(x, t) = f(x) * \Psi_t^\epsilon(x) + g(x) * \Phi_t^\epsilon(x) \quad (2.13)
\]

which \( \epsilon \)-approximation of \( E(x, t) \) in (2.13) for \( \epsilon \to 0 \), \( E^\epsilon(x, t) \to E(x, t) \)

uniformly. Now

\[
E^\epsilon(x, t) = \int_{\mathbb{R}^n} f(r) \Psi_t^\epsilon(x-r) dr + \int_{\mathbb{R}^n} g(r) \Phi_t^\epsilon(x-r) dr
\]

Thus

\[
|E^\epsilon(x, t)| \leq |\Psi_t^\epsilon(x-r)| \int_{\mathbb{R}^n} |f(r)| dr + |\Phi_t^\epsilon(x-r)| \int_{\mathbb{R}^n} |g(r)| dr
\]

\[
\leq \frac{\Omega_p \Omega_q}{(2\pi)^n c^n \epsilon^n} \frac{\Gamma(n) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-n}{2}\right)} M + \frac{\Omega_p \Omega_q}{(2\pi)^n c^n \epsilon^n-1} \frac{\Gamma(n-1) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-n}{2}\right)} N
\]

\[
e^n |E^\epsilon(x, t)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^n c^n} \frac{\Gamma(n) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-n}{2}\right)} M + \frac{\Omega_p \Omega_q}{2(2\pi)^n c^n} \frac{\Gamma(n-1) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-n}{2}\right)} N
\]

where \( M = \int_{\mathbb{R}^n} |f(r)| dr \) and \( N = \int_{\mathbb{R}^n} |g(r)| dr \), since \( f \) and \( g \) are absolutely integrable.

\[
\lim_{\epsilon \to 0} e^n |E^\epsilon(x, t)| \leq \frac{\Omega_p \Omega_q}{2(2\pi)^n c^n} \frac{\Gamma(n) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-n}{2}\right)} = K.
\]

It follows that \( E(x, t) = O(\epsilon^{-n}) \) as \( \epsilon \to 0 \). Where \( E(x, t) \) is an elementary solution for the operator \( L \) defined by (2.3). This proof is complete.
3 Main Results

Theorem 3.1. Given the nonlinear wave equation

\[
\frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box)u(x, t) = f(x, t, u(x, t))
\]  
(3.1)

for \((x, t) \in \mathbb{R}^n \times (0, \infty)\), \(k\) is a positive number and with the following conditions on \(u\) and \(f\) as follows

1. \(u(x, t)\) is the space of continuous function on \(\mathbb{R}^n \times (0, \infty)\).
2. \(f\) satisfies the Lipchitz condition,
   \[|f(x, t, u) - f(x, t, w)| \leq A|u - w|\]
   where \(A\) is constant with \(0 < A < 1\).
3. \(\int_0^\infty \int_{\mathbb{R}^n} |f(x, t, u(x, t))| \, dx \, dt < \infty\) for \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, 0 < t < \infty\) and \(u(x, t)\) is continuous function on \(\mathbb{R}^n \times (0, \infty)\).

Then we obtain the convolution

\[u(x, t) = E(x, t) * f(x, t, u(x, t))\]  
(3.2)

as a unique solution of (3.1) for \(x \in \Omega\) where \(\Omega\) is a compact subset of \(\mathbb{R}^n\) and \(0 \leq t \leq T\) with \(T\) is constant and \(E(x, t)\) is an elementary solution defined by (2.5) and also \(u(x, t)\) is bounded for any fixed \(t > 0\). In particular, if we put \(q = 0\) in (3.1), then (3.1) reduces to the nonlinear wave equation

\[
\frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Delta)u(x, t) = f(x, t, u(x, t))
\]

Proof. Convolving both sides of (3.1) with \(E(x, t)\), that is

\[E(x, t) * \left[ \frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box)u(x, t) \right] = E(x, t) * f(x, t, u(x, t))\]

or

\[\left[ \frac{\partial^2}{\partial t^2} E(x, t) + c^2(\Box)E(x, t) \right] * u(x, t) = E(x, t) * f(x, t, u(x, t)),\]

so

\[\delta(x, t) * u(x, t) = E(x, t) * f(x, t, u(x, t)).\]
Thus
\[ u(x, t) = E(x, t) * f(x, t, u(x, t)) \]
\[ = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r, s) f(x - r, t - s, u(x - r, t - s)) dr ds \]

where \( E(r, s) \) is given by definition (2.4). We next show that \( u(x, t) \) is bounded on \( \mathbb{R}^n \times (0, \infty) \). We have

\[ |u(x, t)| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r, s)||f(x - r, t - s, u(x - r, t - s))| dr ds \]
\[ \leq \frac{2^{2-n}NM(t)}{\pi^{n/2}\Gamma(p/2)\Gamma(q/2)} \]

by condition (3) and (2.6)

where \( N = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |f(x - r, t - s, u(x - r, t - s))| dr ds \). Thus \( u(x, t) \) is bounded on \( \mathbb{R}^n \times (0, \infty) \). To show that \( u(x, t) \) is unique. Suppose there is another solution \( w(x, t) \) of (3.1). We next to show that \( u(x, t) \) is unique. Let \( w(x, t) \) be another solution of (2.1), then

\[ w(x, t) = E(x, t) * f(x, t, w(x, t)) \]

for \( (x, t) \in \Omega_0 \times (0, T] \) the compact subset of \( \mathbb{R}^n \times [0, \infty) \) and \( E(x, t) \) is defined by (2.4). Now, define \( \|u(x, t)\| = \sup_{x \in \Omega_0, 0 < t \leq T} |u(x, t)| \). Now,

\[ |u(x, t) - w(x, t)| = |E(x, t) * f(x, t, u(x, t)) - E(x, t) * f(x, t, w(x, t))| \]
\[ \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r, s)| \cdot |f(x - r, t - s, u(x - r, t - s)) - f(x - r, t - s, w(x - r, t - s))| dr ds \]
\[ \leq A|E(r, s)| \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |u(x - r, t - s) - w(x - r, t - s)| dr ds \]

By the condition (2) of the theorem. Now, for \( (x, t) \in \Omega_0 \times (0, T] \) we have

\[ |u - w| \leq A|E(r, s)||u - w| \int_{0}^{T} ds \int_{\Omega_0} dr \]
\[ = A|E(r, s)| TV(\Omega_0) \|u - w\| \] (3.3)

where \( V(\Omega_0) \) is the volume of the surface on \( \Omega_0 \).

Choose \( A|E(r, s)| TV(\Omega_0) \leq 1 \) or \( A \leq \frac{1}{|E(r, s)| TV(\Omega_0)} \).

Thus from (3.3),

\[ \|u - w\| \leq \alpha\|u - w\| \text{ where } \alpha = A|E(r, s)| TV(\Omega_0) \leq 1. \]
It follows that \( \|u - w\| = 0 \), thus \( u = w \).

That is the solution \( u \) of (2.1) is unique.

Thus \( u(x, t) = w(x, t) \). It follows that the solution \( u(x, t) \) of (3.1) is unique for \( (x, t) \in \Omega \times (0, T) \) where \( u(x, t) \) is defined by (3.2). In particular, if we put \( q = 0 \) in (3.1), then (3.1) reduces to the nonlinear wave equation

\[
\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\Delta) u(x, t) = f(x, t, u(x, t))
\]

which has solution

\[
u(x, t) = E(x, t) * f(x, t, u(x, t))
\]

where \( E(x, t) \) is defined by (2.5) with \( q = 0 \). In particular, if we put \( n = 1 \) and \( q = 0 \) then (3.1) reduces to the solution of the one-dimensional nonlinear wave equation,

\[
\frac{\partial^2}{\partial t^2} u(x, t) + c^2 \frac{\partial^2}{\partial x^2} u(x, t) = f(x, t, u(x, t)).
\]

with the initial conditions

\[
u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = g(x)
\]

where \( f \) and \( g \) are continuous and absolutely integrable for \( x \in \mathbb{R}^n \).

Thus we obtain \( u(x, t) = O(\epsilon^{-1}) * f(x, t, u(x, t)) \) which is a solution of such one-dimensional nonlinear wave equation.

\[\square\]

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References


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