Bicontinuous Maps in Biclosure Spaces

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Abstract
The purpose of this paper is to introduce and study the concept of biclosure spaces. We introduce the notion of bicontinuous maps in biclosure spaces and investigate its behaviour.

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1 INTRODUCTION

J.C. Kelly [6] introduce the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. Closure spaces were introduced by E. Čech [2] and then studied by many authors, see e.g. [3, 4, 7, 8]. In this paper we introduce and study the concept of biclosure spaces. We introduce the concept of bicontinuous maps in biclosure space and characterize their properties.

2 PRELIMINARIES

A map \( u : P(X) \rightarrow P(X) \) defined on the power set \( P(X) \) of a set \( X \) is called a closure operator on \( X \) and the pair \( (X, u) \) is called a closure space if the following axioms are satisfied:

\[(N1) \ u \emptyset = \emptyset,\]
(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator $u$ on a set $X$ is called additive (respectively, idempotent) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is closed in the closure space $(X, u)$ if $uA = A$ and it is open if its complement in $X$ is closed. The empty set and the whole space are both open and closed. A closure space $(Y, v)$ is said to be a subspace of $(X, u)$ if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If $Y$ is closed in $(X, u)$, then the subspace $(Y, v)$ of $(X, u)$ is said to be closed too. Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is said to be continuous if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \rightarrow (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Clearly, if $f : (X, u) \rightarrow (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of $(X, u)$ for every subset $F$ of $(Y, v)$.

Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is said to be closed (resp. open) if $f(F)$ is a closed (resp. open) subset of $(Y, v)$ whenever $F$ is a closed (resp. open) subset of $(X, u)$.

The product of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$, is the closure space $(\prod_{\alpha \in I} X_\alpha, u)$ where $\prod_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets $X_\alpha$, $\alpha \in I$, and $u$ is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in I} X_\alpha \rightarrow X_\alpha$, $\alpha \in I$, i.e., is defined by $uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha \in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \rightarrow (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

**Proposition 2.1.** Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then $F$ is a closed subset of $(X_\beta, u_\beta)$ if and only if $F \times \prod_{\alpha \neq \beta} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

**Proof.** Let $F$ be a closed subset of $(X_\beta, u_\beta)$. Since $\pi_\beta$ is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod_{\alpha \neq \beta} X_\alpha$, hence $F \times \prod_{\alpha \neq \beta} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\alpha \neq \beta} X_\alpha$ be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. Since $\pi_\beta$ is closed, $\pi_\beta(F \times \prod_{\alpha \neq \beta} X_\alpha) = F$ is a closed subset of $(X_\beta, u_\beta)$. \qed
The following statement is evident:

**Proposition 2.2.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces and let \( \beta \in I \). Then \( G \) is an open subset of \( (X_\beta, u_\beta) \) if and only if \( G \times \prod_{\alpha \in I ; \alpha \neq \beta} X_\alpha \) is an open subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

## 3 BICLOSURE SPACES

In this section, we introduce the notion of biclosure spaces and investigate some of its basic properties.

**Definition 3.1.** A biclosure space is a triple \( (X, u_1, u_2) \) where \( X \) is a set and \( u_1, u_2 \) are two closure operators on \( X \).

**Definition 3.2.** A subset \( A \) of a biclosure space \( (X, u_1, u_2) \) is called closed if \( u_1u_2A = A \). The complement of closed set is called open.

Clearly, \( A \) is a closed subset of a biclosure space \( (X, u_1, u_2) \) if and only if \( A \) is both a closed subset of \( (X, u_1) \) and \( (X, u_2) \).

Let \( A \) be a closed subset of a biclosure space \( (X, u_1, u_2) \). The following conditions are equivalent

(i) \( u_1u_2A = A \),

(ii) \( u_1A = A, u_2A = A \).

The following statement is evident:

**Proposition 3.3.** Let \( (X, u_1, u_2) \) be a biclosure space. If \( A \) and \( B \) are closed subsets of \( (X, u_1, u_2) \), then so is \( A \cap B \).

The union of two closed sets need not be closed as can be seen from the following example.

**Example 3.4.** Let \( X = \{1, 2, 3\} \) and define a closure operator \( u_1 \) on \( X \) by \( u_1\emptyset = \emptyset, u_1\{1\} = \{1\}, u_1\{2\} = \{2\}, u_1\{3\} = \{3\} \) and \( u_1\{1, 2\} = u_1\{1, 3\} = u_1\{2, 3\} = u_1X = X \). Define a closure operator \( u_2 \) on \( X \) by \( u_2\emptyset = \emptyset, u_2\{1\} = \{1\}, u_2\{2\} = \{2\}, u_2\{1, 2\} = \{1, 2\} \) and \( u_2\{3\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X \). Then \( \{1\} \) and \( \{2\} \) are closed. But \( \{1\} \cup \{2\} = \{1, 2\} \) is not closed.

**Proposition 3.5.** Let \( (X, u_1, u_2) \) be a biclosure space and let \( u_1 \) and \( u_2 \) be additive. If \( A \) and \( B \) are closed subset of \( (X, u_1, u_2) \), then so is \( A \cup B \).

**Proof.** Since \( u_1u_2A = A \) and \( u_1u_2B = B \), \( u_1u_2A \cup u_1u_2B = A \cup B \). Since \( u_1 \) and \( u_2 \) are additive, \( u_1u_2(A \cup B) = u_1(u_2A \cup u_2B) = u_1u_2A \cup u_1u_2B = A \cup B \). Therefore, \( A \cup B \) is closed. \( \square \)
The following statement is obvious:

**Proposition 3.6.** Let \((X, u_1, u_2)\) be a biclosure space and let \(A \subseteq X\). Then

(i) \(A\) is open if and only if \(A = X - u_1u_2(X - A)\).

(ii) If \(G\) is open and \(G \subseteq A\), then \(G \subseteq X - u_1u_2(X - A)\).

**Definition 3.7.** Let \((X, u_1, u_2)\) be a biclosure space. A biclosure space \((Y, v_1, v_2)\) is called a subspace of \((X, u_1, u_2)\) if \(Y \subseteq X\) and \(v_iA = u_iA \cap Y\) for each \(i \in \{1, 2\}\) and each subset \(A \subseteq Y\).

**Proposition 3.8.** Let \((X, u_1, u_2)\) be a biclosure space and let \((Y, v_1, v_2)\) be a closed subspace of \((X, u_1, u_2)\). If \(F\) is a closed subset of \((Y, v_1, v_2)\), then \(F\) is a closed subset of \((X, u_1, u_2)\).

**Proof.** Let \(F\) be a closed subset of \((Y, v_1, v_2)\). Then \(v_1F = F\) and \(v_2F = F\). Since \(Y\) is both a closed subset of \((X, u_1)\) and \((X, u_2)\), \(u_1F = F\) and \(u_2F = F\). Consequently, \(F\) is both a closed subset of \((X, u_1)\) and \((X, u_2)\). Therefore, \(F\) is a closed subset of \((X, u_1, u_2)\). \(\Box\)

**Proposition 3.9.** Let \(\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}\) be a family of biclosure spaces and let \(\beta \in I\). Then \(F\) is a closed subset of \((X_\beta, u^1_\beta, u^2_\beta)\) if and only if \(F \times \prod\limits_{\alpha \neq \beta} (X_\alpha, u^1_\alpha, u^2_\alpha)\) is a closed subset of \(\prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)\).

**Proof.** Let \(\beta \in I\) and let \(F\) be a closed subset of \((X_\beta, u^1_\beta, u^2_\beta)\). Then \(F\) is a closed subset of \((X_\beta, u^1_\beta)\) and \((X_\beta, u^2_\beta)\), respectively. Since \(\pi_\beta : \prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)\) is continuous, \(\pi_\beta^{-1}(F) = F \times \prod\limits_{\alpha \in I, \alpha \neq \beta} X_\alpha\) is a closed subset of \(\prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha)\).

Similarly, since \(\pi_\beta : \prod\limits_{\alpha \in I} (X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)\) is continuous, \(\pi_\beta^{-1}(F) = F \times \prod\limits_{\alpha \in I, \alpha \neq \beta} X_\alpha\) is a closed subset of \(\prod\limits_{\alpha \in I} (X_\alpha, u^2_\alpha)\). Consequently, \(F \times \prod\limits_{\alpha \in I} X_\alpha\) is a closed subset of \(\prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)\).

Conversely, let \(F \times \prod\limits_{\alpha \in I} X_\alpha\) be a closed subset of \(\prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)\). Then \(F \times \prod\limits_{\alpha \in I} X_\alpha\) is a closed subset of \(\prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha)\) and \(\prod\limits_{\alpha \in I} (X_\alpha, u^2_\alpha)\), respectively. Since \(\pi_\beta : \prod\limits_{\alpha \in I} (X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)\) is closed, \(\pi_\beta(F \times \prod\limits_{\alpha \in I} X_\alpha) = F\) is a closed subset of \((X_\beta, u^1_\beta)\). Similarly, since \(\pi_\beta : \prod\limits_{\alpha \in I} (X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)\) is closed, \(\pi_\beta(F \times \prod\limits_{\alpha \in I} X_\alpha) = F\) is a closed subset of \((X_\beta, u^2_\beta)\). Consequently, \(F\) is a closed subset of \((X_\beta, u^1_\beta, u^2_\beta)\). \(\Box\)
Proposition 3.10. Let \( \{(X_\alpha, u^{1}_\alpha, u^{2}_\alpha) : \alpha \in I \} \) be a family of biclosure spaces and let \( \beta \in I \). Then \( G \) is an open subset of \( (X_\beta, u^{1}_\beta, u^{2}_\beta) \) if and only if \( G \times \prod_{\alpha \neq \beta} X_\alpha \) is an open subset of \( \prod_{\alpha \in I} (X_\alpha, u^{1}_\alpha, u^{2}_\alpha) \).

Proof. Let \( \beta \in I \) and let \( G \) be an open subset of \( (X_\beta, u^{1}_\beta, u^{2}_\beta) \). Then \( X_\beta - G \) is a closed subset of \( (X_\beta, u^{1}_\beta, u^{2}_\beta) \). By Proposition 3.9, \( (X_\beta - G) \times \prod_{\alpha \in I} X_\alpha \) is a closed subset of \( \prod_{\alpha \neq \beta} X_\alpha \). By Proposition 3.9, \( (X_\beta - G) \times \prod_{\alpha \in I} X_\alpha \) is a closed subset of \( \prod_{\alpha \neq \beta} X_\alpha \). Therefore, \( G \times \prod_{\alpha \in I} X_\alpha \) is an open subset of \( \prod_{\alpha \in I} (X_\alpha, u^{1}_\alpha, u^{2}_\alpha) \).

Conversely, let \( G \times \prod_{\alpha \in I} X_\alpha \) be an open subset of \( \prod_{\alpha \in I} (X_\alpha, u^{1}_\alpha, u^{2}_\alpha) \). Then \( \prod_{\alpha \in I} X_\alpha - G \times \prod_{\alpha \in I} X_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u^{1}_\alpha, u^{2}_\alpha) \). But \( \prod_{\alpha \in I} X_\alpha - G \times \prod_{\alpha \in I} X_\alpha = (X_\beta - G) \times \prod_{\alpha \in I} X_\alpha \), hence \( (X_\beta - G) \times \prod_{\alpha \in I} X_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u^{1}_\alpha, u^{2}_\alpha) \). By Proposition 3.9, \( X_\beta - G \) is a closed subset of \( (X_\beta, u^{1}_\beta, u^{2}_\beta) \). Consequently, \( G \) is an open subset of \( (X_\beta, u^{1}_\beta, u^{2}_\beta) \).

4 Bicontinuous Maps

In this section, we introduce the concept of bicontinuous maps in biclosure spaces and study some of their properties.

Definition 4.1. Let \( (X, u_1, u_2) \) and \( (Y, v_1, v_2) \) be biclosure spaces and let \( i \in \{1,2\} \). A map \( f : (X, u_i, u_2) \to (Y, v_1, v_2) \) is called \( i \)-continuous if the map \( f : (X, u_i) \to (Y, v_i) \) is continuous. A map \( f \) is called continuous if \( f \) is \( i \)-continuous for each \( i \in \{1,2\} \).

Definition 4.2. Let \( (X, u_1, u_2) \) and \( (Y, v_1, v_2) \) be biclosure spaces. A map \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is called bicontinuous if the map \( f : (X, u_1) \to (Y, v_2) \) is continuous.

Proposition 4.3. Let \( (X, u_1, u_2) \) and \( (Y, v_1, v_2) \) be biclosure spaces. Then \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is bicontinuous if and only if \( u_1 f^{-1}(B) \subseteq f^{-1}(v_2 B) \) for every \( B \subseteq Y \).

Proof. Let \( B \subseteq Y \). Then \( f^{-1}(B) \subseteq X \). Since \( f \) is bicontinuous, \( f(u_1 f^{-1}(B)) \subseteq v_2 f(f^{-1}(B)) \subseteq v_2 B \). Therefore, \( u_1 f^{-1}(B) \subseteq f^{-1}(v_2 B) \).
Conversely, let \( A \subseteq X \). Then \( f(A) \subseteq Y \). Thus \( u_1 f^{-1}(f(A)) \subseteq f^{-1}(v_2 f(A)) \). Consequently, \( f(u_1 A) \subseteq f(u_1 f^{-1}(f(A))) \subseteq f(f^{-1}(v_2 f(A))) \subseteq v_2 f(A) \). Hence, \( f \) is bicontinuous.

**Proposition 4.4.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. If \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is bicontinuous and \( g : (Y, v_1, v_2) \to (Z, w_1, w_2) \) is 2-continuous, then \( g \circ f : X \to Z \) is bicontinuous.

**Proof.** Let \( A \subseteq X \). Since \( g \circ f(u_1 A) = g(f(u_1 A)) \) and \( f \) is bicontinuous, \( g(f(u_1 A)) \subseteq g(v_2 f(A)) \). Since \( g \) is 2-continuous, \( g(v_2 f(A)) \subseteq w_2 g(f(A)) \). Thus \( g \circ f(u_1 A) \subseteq w_2 g \circ f(A) \). Consequently, \( g \circ f \) is bicontinuous.

**Definition 4.5.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces and let \( i \in \{1, 2\}\). A map \( f : (X, u_i) \to (Y, v_i) \) is called \( i\)-closed (resp. \( i\)-open) if the map \( f : (X, u_i) \to (Y, v_i) \) is closed (resp. open). A map \( f \) is called closed (resp. open) if \( f \) is \( i\)-closed (resp. \( i\)-open) for each \( i \in \{1, 2\} \).

**Definition 4.6.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. A map \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is called biclosed (resp. biopen) if the map \( f : (X, u_1) \to (Y, v_2) \) is closed (resp. open).

**Proposition 4.7.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. If \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is 1-closed and \( g : (Y, v_1, v_2) \to (Z, w_1, w_2) \) is biclosed, then \( g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2) \) is biclosed.

**Proof.** Let \( F \) be a closed subset of \((X, u_1)\). Since \( f \) is 1-closed, \( f(F) \) is a closed subset of \((Y, v_1)\). Since \( g \) is biclosed, \( g(f(F)) \) is a closed subset of \((Z, w_2)\). Hence, \( g \circ f(F) \) is a closed subset of \((Z, w_2)\). Consequently, \( g \circ f \) is biclosed.

**Proposition 4.8.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. Let \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) and \( g : (Y, v_1, v_2) \to (Z, w_1, w_2) \) be maps. Then

(i) If \( g \circ f \) is biclosed and \( f \) is surjective 1-continuous, then \( g \) is biclosed.

(ii) If \( g \circ f \) is biclosed and \( g \) is injective 2-continuous, then \( f \) is biclosed.

**Proof.** (i) Let \( F \) be a closed subset of \((Y, v_1)\). Since \( f \) is 1-continuous, \( f^{-1}(F) \) is a closed subset of \((X, u_1)\). Since \( g \circ f \) is biclosed and \( f \) is surjective, \( g \circ f(f^{-1}(F)) = g(F) \) is a closed subset of \((Z, w_2)\). Hence, \( g \) is biclosed.

(ii) Let \( F \) be a closed subset of \((X, u_1)\). Since \( g \circ f \) is biclosed, \( g \circ f(F) \) is a closed subset of \((Z, w_2)\). Since \( g \) is 2-continuous and injective, \( g^{-1}(g \circ f(F)) = f(F) \) is a closed subset of \((Y, v_2)\). Therefore, \( f \) is biclosed.

The following statement is evident:
Proposition 4.9. Let \( \{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\} \) be a family of biclosure spaces. Then for each \( \beta \in I \), the projection map \( \pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \rightarrow (X_\beta, u^1_\beta, u^2_\beta) \) is closed.

Proposition 4.10. Let \( \{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\} \) and \( \{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\} \) be families of biclosure spaces. For each \( \alpha \in I \), let \( f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \rightarrow (Y_\alpha, v^1_\alpha, v^2_\alpha) \) be a surjection and let \( f : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha) \) be defined by \( f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I} \). Then \( f \) is biclosed if and only if \( f_\alpha \) is biclosed for each \( \alpha \in I \).

Proof. Let \( \beta \in I \) and let \( F \) be a closed subset of \( (X_\beta, u^1_\beta) \). Then \( F \times \prod_{\alpha \in I} X_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \). Since \( f \) is biclosed, \( f(F \times \prod_{\alpha \in I} X_\alpha) \) is a closed subset of \( \prod_{\alpha \in I} (Y_\alpha, v^2_\alpha) \). But \( f(F \times \prod_{\alpha \in I} X_\alpha) = f_\beta(F) \times \prod_{\alpha \in I} Y_\alpha \), hence \( f_\beta(F) \times \prod_{\alpha \in I} Y_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (Y_\alpha, v^2_\alpha) \). By Proposition 2.1, \( f_\beta(F) \) is a closed subset of \( (Y_\beta, v^2_\beta) \). Hence, \( f_\beta \) is biclosed.

Conversely, let \( f_\beta \) be biclosed for each \( \beta \in I \). Suppose that \( f \) is not biclosed. Then there exists a closed subset \( F \) of \( \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \) such that \( \prod_{\alpha \in I} v^2_\alpha \pi_\beta(f(F)) \nsubseteq f(F) \). Therefore, there exists \( \beta \in I \) such that \( v^2_\beta f_\beta(\pi_\beta(F)) \nsubseteq f_\beta(\pi_\beta(F)) \). But \( \pi_\beta(F) \) is a closed subset of \( (X_\beta, u^1_\beta) \) and \( f_\beta \) is biclosed, \( f_\beta(\pi_\beta(F)) \) is a closed subset of \( (Y_\beta, v^2_\beta) \). This is a contradiction.

The following statement is evident:

Proposition 4.11. Let \( \{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\} \) be a family of biclosure spaces. Then for each \( \beta \in I \), the projection map \( \pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \rightarrow (X_\beta, u^1_\beta, u^2_\beta) \) is continuous.

Proposition 4.12. Let \( (X, u_1, u_2) \) be a biclosure space, \( \{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\} \) be a family of biclosure spaces and \( f : (X, u_1, u_2) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha) \) be a map. Then \( f \) is bicontinuous if and only if \( \pi_\alpha \circ f \) is bicontinuous for each \( \alpha \in I \).

Proof. Let \( f \) be bicontinuous. Since \( \pi_\alpha \) is 2-continuous for each \( \alpha \in I \), \( \pi_\alpha \circ f \) is bicontinuous for each \( \alpha \in I \).

Conversely, let \( \pi_\alpha \circ f \) be bicontinuous for each \( \alpha \in I \). Suppose that \( f \) is not bicontinuous. Then there exists a subset \( A \) of \( X \) such that \( f(u_1A) \nsubseteq \prod_{\alpha \in I} v^2_\alpha \pi_\alpha(f(A)) \). Therefore, there exists \( \beta \in I \) such that \( \pi_\beta(f(u_1A)) \nsubseteq v^2_\beta \pi_\beta(f(A)) \). This is contradict the bicontinuity of \( \pi_\beta \circ f \). Consequently, \( f \) is bicontinuous.
Proposition 4.13. Let \( \{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\} \) and \( \{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\} \) be families of biclosure spaces. For each \( \alpha \in I \), let \( f_\alpha : (X_\alpha, u^1_\alpha) \rightarrow (Y_\alpha, v^1_\alpha) \) be a map and let \( f : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha) \) be defined by \( f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I} \). Then \( f \) is bicontinuous if and only if \( f_\alpha \) is bicontinuous for each \( \alpha \in I \).

Proof. Let \( f \) be bicontinuous, let \( \beta \in I \) and let \( A \subseteq X_\beta \). Then

\[
f_\beta(u^1_\beta A) = \pi_\beta \left( f_\beta(u^1_\beta A) \times \prod_{\alpha \neq \beta \atop \alpha \in I} f_\alpha(u^1_\alpha X_\alpha) \right)
= \pi_\beta \left( f \left( u^1_\beta A \times \prod_{\alpha \neq \beta \atop \alpha \in I} u^1_\alpha X_\alpha \right) \right)
= \pi_\beta \left( f \left( \prod_{\alpha \in I} u^1_\alpha \pi_\alpha \left( A \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \right) \right) \right)
\subseteq \pi_\beta \left( \prod_{\alpha \in I} v^2_\alpha \pi_\alpha \left( f \left( A \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \right) \right) \right)
= \pi_\beta \left( \prod_{\alpha \in I} v^2_\alpha \pi_\alpha \left( f_\beta(A) \times \prod_{\alpha \neq \beta \atop \alpha \in I} f_\alpha(X_\alpha) \right) \right)
= \pi_\beta \left( v^2_\beta f_\beta(A) \times \prod_{\alpha \neq \beta \atop \alpha \in I} v^2_\alpha f_\alpha(X_\alpha) \right)
= v^2_\beta f_\beta(A).
\]

Hence, \( f_\beta \) is bicontinuous.

Conversely, let \( f_\alpha \) be bicontinuous for each \( \alpha \in I \) and let \( A \subseteq \prod_{\alpha \in I} X_\alpha \). Then

\[
f(\prod_{\alpha \in I} u^1_\alpha \pi_\alpha(A)) = \prod_{\alpha \in I} f_\alpha(\prod_{\alpha \in I} u^1_\alpha \pi_\alpha(A))
= \prod_{\alpha \in I} f_\alpha(u^1_\alpha \pi_\alpha(A))
\subseteq \prod_{\alpha \in I} v^2_\alpha f_\alpha(\pi_\alpha(A))
= \prod_{\alpha \in I} v^2_\alpha \pi_\alpha(f(A))
\]

Therefore \( f \) is bicontinuous. \( \square \)


References


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