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On Annihilator of Fuzzy Subsets of Modules

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Abstract

The correspondence between certain ideals and submodules arising from annihilation plays a vital role in decomposition theory. In this paper our attempt is to fuzzify the concept of annihilators of subsets of modules. We investigate certain characteristics of fuzzy annihilators of fuzzy subsets of modules. Using the concept of fuzzy annihilators, prime fuzzy sub module and fuzzy annihilator ideals are defined and various related results are established.

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Introduction

In the trajectory of stupendous growth of fuzzy set theory, fuzzy algebra has become an important area of research. In 1971, A. Rosenfield [6] used the concept of fuzzy set theory due to Zadeh [10] in abstract algebra and opened up a new insight in the field of Mathematical science. Since then the study of fuzzy algebraic structure has been pursued in many directions such as semigroups, groups, rings, semirings, near-rings and so on. Fuzzy sub modules of a module M over a ring R were first introduced by Negoita and Ralescu [4]. Consequently Pan [5] studied fuzzy finitely generated modules and fuzzy quotient modules. The study of fuzzy submodules was also carried out by Zahedi [10], Sidky [8], Mukherjee et al [3], and Saikia et al [7]. In the theory of rings and modules there is a correspondence between certain ideals of a ring R and submodules of an R-module that arise from annihilation. The submodules obtained using annihilation, which correspond to prime ideals play an important role in decomposition theory and Goldie like structures. Fuzzification of such crisp sets leads us to structures that can be termed as fuzzy prime submodules. Our attempt is to fuzzify the concept of annihilators of modules. We define annihilator of a fuzzy subset of a module using the concept of residual quotients and investigate various characteristics of it. This concept will help us to explore and investigate various facts about the fuzzy aspects of associated primes, Goldie like structures and singular ideals.

2. Basic definitions and notations

Throughout this paper R denotes a commutative ring with unity and M denotes an R-module. The zero elements of R and M are 0 and θ respectively. The class of fuzzy subsets of X is denoted by $\mathbf{[0, 1]^X}$.

Let $\mu\in[0, 1]^R$. Then μ is called a *fuzzy ideal* of R if it satisfies

(i) $\mu(x\text{-}y) \geq \mu(x) \wedge \mu(y) \quad \forall\ x, y \in R$

(ii) $\mu(xy) \geq \mu(x) \vee \mu(y) \quad \forall\ x, y \in R$.

The class of all fuzzy ideals of R is denoted by **FI(R).** Let $\mu\in[0, 1]^M$ Then μ is called a *fuzzy sub module* of M if it satisfies:

(i) $\mu(\theta) = 1$

(ii) $\mu(x\text{-}y) \geq \mu(x) \wedge \mu(y) \quad \forall\ x, y \in M$

(iii) $\mu(rx) \geq \mu(x) \quad \forall r\in R, x \in M$.

The class of fuzzy sub modules is denoted by **F(M).** Let $\mu\in[0, 1]^X$ .Then a *fuzzy point* x_t, $x \in X$, $t \in (0,1]$ is defined as the fuzzy subset x_t of X such that $x_t(x) = t$ and $x_t(y) = 0$ for all $y\in X - \{x\}$. We write $x_t \in \mu$ if and only if $x \in\mu_t$.

Let $\mu, \sigma \in[0, 1]^M$. Then *sum* of μ and σ is defined as

$(\mu + \sigma)(x) = \vee \{\mu(y) \wedge \sigma(z) \mid y, z \in M, x = y + z\}$.

Let $\mu\in [0, 1]^M$ and $\sigma\in [0, 1]^R$ then the *product* of μ and σ is defined as

$(\sigma\mu)(x) = \vee \{\sigma(r) \wedge \mu(m) \mid r \in R, m \in M, x = rm\}$.

Let $\mu, \sigma\in [0, 1]^R$ Then the *product* of μ and σ is defined as

$(\sigma\mu)(x) = \vee \{\mu(y) \wedge \sigma(z) \mid y, z \in R, x = yz\}$.

By μ^2 and μ^n we mean $\mu\mu$ and $\mu\mu^{n-1}$ respectively. Let $\mu \in FI(R)$. Then μ is called a *fuzzy prime ideal* of R if for any $\sigma, \gamma \in F(R)$, $\sigma\gamma \subseteq \mu$ implies $\sigma \subseteq \mu$ or $\gamma \subseteq \mu$.

Let $\mu \in [0, 1]^M$. then *annihilator of μ*, denoted by ann(μ) is defined as $\text{ann}(\mu) = \cup \{\eta \mid \eta\in [0,1]^R , \eta\mu \subseteq \chi_\theta\}$. It is seen that $\text{ann}(\mu) \in [0,1]^R$. We define $\text{ann}(\chi_\theta)$ as $\text{ann}(\chi_\theta) = \chi_R$

Let $\mu\in [0, 1]^M$ is Then μ is said to be a *fuzzy faithful submodule* if $\text{ann}(\mu) = \chi_0$.

Let $\mu \in [0, 1]^R$ .Then μ is called a *fuzzy dense ideal* if $\text{ann}(\mu) = \chi_0$

Let $\sigma \in [0, 1]^R$ Then the fuzzy ideal of the form ann(σ) of R is called a *fuzzy annihilator ideal*. Thus μ is a fuzzy annihilator ideal if and only if $\mu = \text{ann}(\sigma)$ for some fuzzy subset σ of R with $\sigma(0) = 1$.

3. Preliminaries

Now we present some preliminary results that are needed in the sequel.
Lemma 3.1: [8] Let $\mu \in [0, 1]^M$. Then the level subset $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is a sub module of M if and only if μ is a fuzzy sub module of M.
Lemma 3.2: [1] If $\mu \in FI(R)$ then μ is a fuzzy prime ideal of R if and only if μ_t, $t \in \text{Im}\, \mu$ is a prime ideal of R
Lemma 3.3 :[2]Let $\mu_i \in FI(R)$, $i \in I$ Then $\bigcap_{i \in I} \mu_i \in FI(R)$.
Lemma 3.4 : [2] If $\mu_i \in F(M)$, $i \in I$ where $|I| > 1$ then $\sum_{i \in I} \mu_i \in F(M)$.and
Lemma 3.5: [2] If $\sigma \in FI(R)$; $\mu, \gamma \in F(M)$ then $\sigma(\mu + \gamma) \subseteq \sigma\mu + \sigma\gamma$.
The following results can be established in the same line as in section 4.5 of [2]
Lemma 3.6: If $\mu \in [0, 1]^M$ then $\chi_0 \subseteq \text{ann}(\mu)$.
Lemma 3.7: Let $\mu \in [0, 1]^M$. Then $\text{ann}(\mu) = \cup \{r_\alpha \mid r \in R, \alpha \in [0,1], r_\alpha\mu \subseteq \chi_\theta\}$.
Lemma 3.8: Let $\mu \in [0, 1]^M$. Then $\text{ann}(\mu)\mu \subseteq \chi_\theta$. If $\mu(\theta) = 1$, then $\text{ann}(\mu)\mu = \chi_\theta$. Hence if $\mu \in F(M)$ then $\text{ann}(\mu)\mu = \chi_\theta$.
Lemma 3.9: Let $\sigma \in [0, 1]^R$ and $\mu \in [0, 1]^M$. then $\sigma\mu \subseteq \chi_\theta$, if and only if $\sigma \subseteq \text{ann}(\mu)$. If $\sigma(0) = 1$ and $\mu(\theta) = 1$, then $\sigma\mu = \chi_\theta$ if and only if if $\sigma \subseteq \text{ann}(\mu)$.
Lemma 3.10: Let $\mu \in [0, 1]^M$. Then $\text{ann}(\mu) = \cup \{\sigma \mid \sigma \in FI(R), \sigma\mu \subseteq \chi_\theta\}$.
Lemma 3.11: Let $\mu_i \in [0, 1]^M$, $i \in I$, then $.\text{ann}(\bigcup_{i\in I} \mu_i) = \bigcap_{i \in I} \text{ann}(\mu_i)$

4. Main Results

Now we present the main results.
Theorem 4.1: Let $\mu, \sigma \in [0, 1]^M$. If $\mu \subseteq \sigma$, then $\text{ann}(\sigma) \subseteq \text{ann}(\mu)$.
Proof: Let $\gamma \in [0, 1]^R$. Then for $r \in R$, $m \in M$, $\gamma(r) \wedge \mu(m) \leq \gamma(r) \wedge \sigma(m)$. This implies
$\vee \{\gamma(r) \wedge \mu(m) \mid r \in R, m \in M, x = rm\} \leq \vee \{\gamma(r) \wedge \sigma(m) \mid r \in R, m \in M, x = rm\}$.
So $(\gamma\mu)(x) \leq (\gamma\sigma)(x)$, $\forall x \in M$. Hence $\gamma\mu \subseteq \gamma\sigma$. Therefore $\gamma\sigma \subseteq \chi_\theta$ implies $\gamma\mu \subseteq \chi_\theta$. Hence
$\{\gamma \mid \gamma \in F(R), \gamma\sigma \subseteq \chi_\theta\} \subseteq \{\gamma \mid \gamma \in F(R), \gamma\mu \subseteq \chi_\theta\}$. So $\text{ann}(\sigma) \subseteq \text{ann}(\mu)$.
Theorem 4.2: Let $\mu, \sigma \in [0, 1]^M$. Then the following conditions are equivalent:

(i) $\text{ann}(\gamma) = \text{ann}(\mu)$, for all $\gamma \subseteq \mu, \gamma \neq \chi_\theta$.

(ii) $\gamma\sigma \subseteq \chi_\theta$ implies $\gamma\mu \subseteq \chi_\theta$, for all $\gamma \subseteq \mu$, $\gamma \neq \chi_\theta$, $\sigma \in [0, 1]^R$

Proof: Let $\gamma\sigma \subseteq \chi_\theta$. Then by lemma 3.9, $\sigma \subseteq \text{ann}(\gamma) = \text{ann}(\mu)$
Again by lemma 3.9, $\sigma\mu \subseteq \chi_\theta$.
Also using lemma 3.8 we get

$$\text{ann}(\gamma)\gamma \subseteq \chi_\theta.$$

So (ii) implies $\text{ann}(\gamma)\mu \subseteq \chi_\theta$, where $\gamma \subseteq \mu$ and $\gamma \neq \chi_\theta$
By lemma 3.9 we have $\text{ann}(\gamma) = \text{ann}(\mu)$ .Also $\gamma \subseteq \mu$ implies $\text{ann}(\mu) \subseteq \text{ann}(\gamma)$, by theorem 4.1. So $\text{ann}(\gamma) \subseteq \text{ann}(\mu)$.
Corollary 4.3: If $\mu(\theta) = 1$, $\gamma(\theta) = 1$, then the following conditions are equivalent:

(i) $ann(\gamma) = ann(\mu)$, for all $\gamma \subseteq \mu$, $\gamma \neq \chi_\theta$.

(ii) $\sigma\gamma = \chi_\theta$ implies $\sigma\mu = \chi_\theta$, for all $\gamma \subseteq \mu$, $\gamma \neq \chi_\theta$, $\sigma \in [0, 1]^R$

Theorem 4.4: Let $\mu \in [0, 1]^M$ be faithful. If R is non zero then $\mu \neq \chi_\theta$.

Proof: μ faithful implies $ann(\mu) = \chi_0$.

If $\mu = \chi_\theta$, then $ann(\mu) = ann(\chi_\theta) = \chi_R$. This implies $\chi_0 = \chi_R$. So R = {0}, which is a contradiction. Therefore $\mu \neq \chi_\theta$.

Theorem 4.5: Let $\mu \in [0, 1]^R$ with $\mu(0) = 1$. Then $\mu \subseteq ann(ann(\mu))$ and $ann(ann(ann(\mu))) = ann(\mu)$.

Proof: If μ is a fuzzy subset of R module R then by lemma 3.8 we have, $ann(\mu)\mu = \chi_0$. Now by lemma 3.9, $\mu \subseteq ann(ann(\mu))$.
This implies $ann(ann(ann(\mu))) \subseteq ann(\mu)$. Also $ann(\mu) \subseteq ann(ann(ann(\mu)))$.
So $ann(ann(ann(\mu))) = ann(\mu)$.

Theorem 4.6: If $\mu, \gamma \in F(M)$, then $ann(\mu+ \gamma) = ann(\mu) \cap ann(\gamma)$.

Proof: Let $\mu, \gamma \in F(M)$, implies $\mu + \gamma \in F(M)$. $\mu \subseteq \mu + \gamma$ and $\gamma \subseteq \mu + \gamma$ imply $ann(\mu + \gamma) \subseteq ann(\mu) \cap ann(\gamma)$. Also

$$\begin{aligned}ann(\mu)\cap ann(\gamma) &= (\cup \{\sigma_1 \mid \sigma_1 \in FI(R), \sigma_1\mu \subseteq \chi_\theta\}) \cap (\cup \{\sigma_2 \mid \sigma_2 \in FI(R), \sigma_2\gamma \subseteq \chi_\theta\})\\ &= \cup \{\sigma_1\cap\sigma_2 \mid \sigma_1, \sigma_2 \in FI(R), \sigma_1\mu \subseteq \chi_\theta, \sigma_2\gamma \subseteq \chi_\theta\}\\ &\subseteq \cup \{\sigma \mid \sigma = \sigma_1\cap\sigma_2 \in FI(R), \sigma\mu \subseteq \chi_\theta, \sigma\gamma \subseteq \chi_\theta\}\\ &\subseteq \cup \{\sigma \mid \sigma \in FI(R), \sigma(\mu + \gamma) \subseteq \chi_\theta\}\\ &= ann(\mu + \gamma)\end{aligned}$$

Theorem 4.7: If μ is a fuzzy ideal of a semi prime ring R then $\mu \cap ann(\mu) = \chi_0$ and $\mu + ann(\mu)$ is a fuzzy dense ideal of R.

Proof: Since $\mu \cap ann(\mu) \subseteq \mu$, $\mu \cap ann(\mu) \subseteq ann(\mu)$, so $(\mu \cap ann(\mu))^2 \subseteq \mu ann(\mu) \subseteq \chi_0$. Now R is a semi prime ring and it implies 0 is a semi prime ideal of R. So χ_0 is a fuzzy semi prime ideal of R. Also $(\mu \cap ann(\mu))^2 \subseteq \chi_0$ implies $\mu \cap ann(\mu) = \chi_0$.

Hence $ann(\mu + ann(\mu)) = ann(\mu) \cap ann(ann(\mu)) = \chi_0$ proving thereby $\mu + ann(\mu)$ is a fuzzy dense ideal of R.

Theorem 4.8: Let μ be a non zero fuzzy ideal of a prime ring R with $\mu(0) = 1$.
Then μ is a fuzzy dense ideal of R.

Proof: $\mu ann(\mu) = \chi_0$ implies $ann(\mu) = \chi_0$ or $\mu = \chi_0$. But $\mu \neq \chi_0$, so $ann(\mu) = \chi_0$.

In view of theorem 4.5 it follows that μ is an annihilator ideal of R implies $ann(ann(\mu)) = \mu$.

Theorem 4.9 : The annihilator ideals in a semi prime ring form a complete Boolean algebra $\mathcal{B}$ with intersection as infimum and ann as complementation.

Proof: Since $\bigcap_{i\in I} ann(\mu_i) = ann(\sum_{i\in I} \mu_i)$, so any intersection of annihilator ideals is a fuzzy annihilator ideal. Hence these ideals form a complete semi-lattice with intersection as infimum. To show that they form a Boolean algebra it remains to show that: $\mu \cap ann(\sigma) = \chi_0$ if and only if $\mu \subseteq \sigma$, for annihilator ideals μ and σ.
If $\mu \subseteq \sigma$, then $\mu \cap ann(\sigma) \subseteq \sigma \cap ann(\sigma) = \chi_0$.

Conversely, let $\mu \cap ann(\sigma) = \chi_0$. Now $\mu ann(\sigma) \subseteq \mu \cap ann(\sigma) = \chi_0$.
This implies $\mu \subseteq ann(ann(\sigma)) = \sigma$.

Theorem 4.10: Let M be a non zero R module. Suppose that there exists a fuzzy ideal μ maximal among the annihilators of non zero fuzzy sub modules of M. Then μ is a fuzzy prime ideal of R.

Proof: There is a fuzzy sub module σ ($\neq\chi_\theta$) of M such that $\mu = ann(\sigma)$. Suppose that α and β and are fuzzy ideals of R properly containing μ (i.e. $\mu \subset \alpha$, $\mu \subset \beta$) such that $\alpha\beta \subseteq \mu$. If $\beta\sigma = \chi_\theta$, then $\beta \subseteq ann(\sigma) = \mu$, which is a contradiction to our supposition. So $\beta\sigma \neq \chi_\theta$. Now $\alpha\beta \subseteq \mu$ implies $\alpha(\beta\sigma) \subseteq \mu\sigma = ann(\sigma)\sigma = \chi_\theta$. So $\alpha \subseteq ann(\beta\sigma)$. Hence $\mu \subseteq ann(\beta\sigma)$. This contradicts the maximality of μ. So μ is a fuzzy prime ideal of R.

If $\mu\in$ F(M), $\mu \neq \chi_\theta$, satisfying one (hence both) of the conditions of theorem 4.2 then μ is called a *fuzzy prime sub module* of M..

Theorem 4.11: If μ is a fuzzy prime sub module of M then ann(μ) is a fuzzy prime ideal of R.

Proof: Let μ be a prime fuzzy sub module and $\alpha\beta \subseteq ann(\mu)$ where β is not contained in ann(μ). Then $\chi_\theta \neq \beta\mu \subseteq \mu$. Now $\alpha\beta \subseteq ann(\mu)$ implies $(\alpha\beta)\mu \subseteq ann(\mu)\mu = \chi_\theta$. So $\alpha \subseteq ann(\beta\mu) = ann(\mu)$, as μ is prime. Hence ann(μ) is prime.

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