On Residual Transcendental Extensions
of $v$ to $K(x_1, \ldots, x_n)$

Figen Öke

Trakya University, Department of Mathematics, 22030 Edirne, Turkey
figenoke@gmail.com

Abstract

An r.t. extension of a valuation $v$ on $K$ to a rational function field with $n$ variables is defined by using r.t. extensions of $v$ to a rational function field with one variable and some properties of this valuation are investigated.

Mathematics Subject Classification: 12J10, 12F20, 12J20

Keywords: extensions of valuations, residual transcendental extensions

1 Introduction

Residual transcendental extensions of $v$ to $K(x)$ are described by N. Popescu, V. Alexandru and A. Zaharescu in 1988, 1990. All valuations on $K(x)$ are classified by them in 1990. Certain residual transcendental and residual algebraic extensions of $v$ to $K(x, y)$ are defined by F. Öke and H. İşcan in 2002. In this paper a residual transcendental extension of $v$ to $K(x_1, \ldots, x_n)$ is defined using residual transcendental extensions of $v$ to $K(x_i)$ for $i = 1, \ldots, n$. Certain properties of residual transcendental extensions of $v$ to a rational function field with one variable are generalized for rational function field with $n$ variables.

2 Preliminaries and Some Notations

Throughout this paper, $v$ is a valuation of a field $K$ with value group $G_v$, valuation ring $O_v$ and residue field $k_v$, $K$ is an algebraic closure of $K$, $\overline{v}$ is the unique extension of $v$ to $\overline{K}$, the value group of $\overline{v}$ is the divisible closure
of $G_v$ i.e. $\overline{G_v} = G_v = QG_v$ is the smallest divisible group which contains $G_v$ and the residue field $\overline{k_v} = k_\overline{v}$ of $\overline{v}$ is the algebraic closure of $k_v$. Let $L$ be an extension of a field $K$. If $v'$ is an extension of $v$ to $L$ then $k_v$ will be identified canonically with a subfield of $k_{v'}$ and $G_v$ with a subgroup of $G_{v'}$. Let $K$ and $M$ be subfields of a field $L$ and $u$ be a valuation of $L$. If the restriction of $u$ to $K$ is a valuation $v$ and the restriction of $u$ to $L$ is valuation $w$ then $u$ is called common extension of $v$ and $w$ to $L$. $K(x)$ and $(x_1, ..., x_n)$ are rational function fields over $K$ with one and $n$ variables respectively. For any $b \in O_v$, $b^*$ denotes the natural image of $b$ in $k_v$. If $a_1, ..., a_n \in \overline{K}$, then the restriction of $\overline{v}$ to $K(a_1, ..., a_n)$ will be denoted by $v_{a_1, ..., a_n}$.

$w$ is called residual transcendental (r.t.) extension of $v$ if $k_w/k_v$ is a transcendental extension. For each $F = \sum_{i=1}^{n} a_i x^i \in K[x]$, $w$ defined as $w(\sum_{i} a_i x^i) = \min(v(a_i))$ is a r. t. extension of $v$ to $K(x)$. It is called as Gaussian extension of $v$. Its value group is $G_w = G_v$ and residue field is $k_w = k_v(x^*)$, where $x^*$ trans / $k_v$.

An element $(a, \delta)$ of $\overline{K} \times G$ where $G$ is an ordered abelian group which contains $G_v$ is usually called a pair. A pair $(a, \delta)$ is called minimal with respect to $K$ if for every $b \in \overline{K}$ such that $[K(b) : K] < [K(a) : K]$, one has $\overline{v}(a - b) < \delta$. If $w$ is an r.t. extension of $v$ to $K(x)$ then there exists a minimal pair $(a, \delta) \in \overline{K} \times G_v$ respect to $K$ where $a$ is separable over $K$ and $w$ is defined as follows: Let $f = \text{Irr}(a, K)$ be a minimal polynomial of $a$ respect to $K$ and $\gamma = w(f)$. If $F \in K[x]$, $F = F_0 + F_1 f + ... + F_n f^n$, $\deg F_i < \deg f$, $i = 0, ..., n$ then $w(F) = \inf(v_a(F_i(a)) + i\gamma)$. Let $e$ be the smallest non-zero positive integer such that $e\gamma \in G_{v_a}$. Then $G_w = G_{v_a} + Z\gamma$, $[G_w : G_v] = e[G_{v_a} : G_v]$. Let $h \in K[x]$ such that $\deg h < \deg f$, $v_a(h(a)) = e\gamma$. Then $r = f^e/h$ is an element of $O_w$ of the smallest order such that $r^* \in k_w$ is transcendental over $k_v$. Thus the field $k_{v_a}$ can be identified canonically with the algebraic closure of $k_v$ in $k_w$ and $k_w = k_{v_a}(r^*)$. For any $r \in K(x)$, $r \notin K$, $\deg r$ is defined as $\deg r = [K(x) : K(r)]$. If $w$ is a r.t. extension of $v$ to $K(x)$ defined as above $\deg(w/v)$ is the least natural number such that there exists $r \in O_w$ of degree $n$ and $r^*$ is transcendental over $k_v$. The ramification index of $G_w$ over $G_v$ is denoted by $e(w/v) = [G_w : G_v]$ and the residue degree of $w$ over $v$ is $f(w/v) = [k_{v_a} : k_v]$ which is degree of algebraic closure of $k_v$ in $k_w$. 
3 Results

Let \( w_i \) be a r.t. extension of \( v \) to \( K(x_i) \) defined by a minimal pair \((a_i, \delta_i) \in \overline{K} \times G_\delta \) and \( f_i = \text{Irr}(a_i, K) \). \( \gamma_i = w_i(f_i) \), \( e_i \) be the smallest number such that \( e_i \gamma_i \in G_{\upsilon_1} \) where \( v_\upsilon \) is the restriction of \( \upsilon \) to \( K(a_i) \), \( h_i \in K[x_i] \) such that \( \deg h_i < \deg f_i \) and \( v_\upsilon(h(a_i)) = e_i \gamma_i \) for \( i = 1, ..., n \) and such that \( r_i^* \text{ trans}/k_\upsilon \).

Each polynomial \( F \in K[x_1, ..., x_n] \) can be written uniquely as:

\[
F = \sum_{i_1, ..., i_n} F_{i_1}^{i_1} ... F_{i_n}^{i_n}, \text{ deg}_x F_{i_1}^{i_1} ... F_{i_n}^{i_n} < \deg f_i, \ i = 1, ..., n.
\]

Define

\[
u(F) = \inf_{i_1, ..., i_n} (v_{a_1} ... a_n (F_{i_1}^{i_1} ... i_n (a_1, ..., a_n))) + i_1 \gamma_1 + ... + i_n \gamma_n \tag{1}\]

It is easily seen that \( \nu \) satisfies all valuation conditions on \( K[x_1, ..., x_n] \) and it can be uniquely extended to \( K(x_1, ..., x_n) \).

**Proposition 2.1:** Let \( u \) be a valuation which is defined in (1). \( u \) is a r.t. extension of \( v \) to \( K(x_1, ..., x_n) \).

1. If \( F(x_1, ..., x_n) \in K[x_1, ..., x_n] \) and \( \deg_x F(x_1, ..., x_n) < \deg f_i \) for \( i = 1, ..., n \) then \( u(F(x_1, ..., x_n)) = \bar{\nu}(F(a_1, ..., a_n)) \) and \( G_w = G_{v_{a_1} ..., a_n} + \mathbb{Z} \gamma_1 + ... + \mathbb{Z} \gamma_n \).

2. \( k_{v_{a_1} ..., a_n} \) is the algebraic closure of \( k_\upsilon \) in \( k_u \) and \( k_u = k_{v_{a_1} ..., a_n} (r_1^*, ..., r_n^*) \) where \( k_u = k_{v_{a_1} ..., a_n} (r_1^*, ..., r_n^*) \) for \( i = 1, ..., n \).

**Proof:** \( w_i \) which is the restriction of \( u \) to \( K(x_i) \) for \( i = 1, ..., n \) is a r.t. extension of \( u \) to \( K(x_i) \) defined by the minimal pair \((a_i, \delta_i) \in \overline{K} \times G_\delta \). \( u \) is the common extension of \( w_i \) to \( K(x_1, ..., x_n) \) for \( i = 1, ..., n \).

1. According to Theo 2.1 of [1] if \( g(x_i) \in K[x_1] \) and \( \deg g < \deg f_i \) then

\[
w_i(g(x_i)) = \bar{\nu}(g(a_i)).
\]

Therefore if \( \deg_x F(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n) < \deg f_i \) then \( u'_i(F(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., x_n)) = \bar{\nu}(F(a_1, ..., a_{i}, ..., a_n)) \) where \( u'_i \) is the restriction of \( u' \) which is the fixed extension of \( u \) to \( \overline{K}(x_1, ..., x_n) \) for

\( i = 1, ..., n \). Hence if \( \deg_x F(x_1, ..., x_n) < \deg f_i \) for \( i = 1, ..., n \) then

\[
u(x_1, ..., x_n) = \bar{\nu}(F(a_1, ..., a_n)) = v_{a_1} ..., a_n (F(a_1, ..., a_n)) \text{ and then}
\]

\[
G_w = G_{v_{a_1} ..., a_n} + \mathbb{Z} \gamma_1 + ... + \mathbb{Z} \gamma_n.
\]

2. Since \( e_i \gamma_i \in G_{v_{a_i}} \) there exists a polynomial \( h_i \in K[x_i] \), \( \deg h_i < \deg f_i \) such that \( w_i(h_i) = v_{a_i} (h(a_i)) \) and \( r_i^* \text{ trans}/k_{v_{a_i}} \) for \( i = 1, ..., n \). Let \( w_{a_i} \) be
the restriction of $u'$ to $K(x_1, \ldots, x_{i-1}, a_i)$ and $u_i$ is the common extension of $w_1, \ldots, w_i$ to $K(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$. $u_1$ is the r.t. extension of $v$ to $K(x_1)$ defined by the minimal pair $(a_1, \delta_1) \in \overline{K} \times G_v$ and $u_i$ is the r.t. extension of $u_{i-1}$ defined by the minimal pair $(a_i, \delta_i) \in \overline{K} \times G_v$ for $i = 2, \ldots, n$. Since $k_{w_1} = k_{v_{u_1}}$, $r_1 \text{trans} / k_{v_{u_1}}$, $k_{u_1} = k_{v_{u_1}}(r_1)$ and $k_{w_i} = k_{w_{u_{i-1}}}, r_i \text{trans} / k_{v_{u_i}}$ for $i = 2, \ldots, n$ then it is obtained that to the algebraic closure of $k_v$ in $k_u$ is $k_{v_{u_1, \ldots, u_n}}$ and $k_u = k_{v_{u_1, \ldots, u_n}}(r_1^*, \ldots, r_n^*)$ by the induction.

**Definition 2.2:** Let $u$ be an extension of $v$ to $K(x_1, \ldots, x_n)$ defined in (1). Then degree of $u/v$ is defined as;

$$D(u/v) = [K(x_1, \ldots, x_n) : K(r_1, \ldots, r_n)]$$

Moreover the ramification index of $u/v$ is defined as; $E = E(u/v) = [G_u : G_v]$ and the residue degree of $u/v$ is defined as, $F = F(u/v) = [k_{v_{u_1, \ldots, u_n}} : k_v]$ which is the degree of the algebraic closure of $k_v$ in $k_u$ over $k_v$.

**Theorem 2.3:** Let $u$ be the r.t. extension of $v$ to $K(x_1, \ldots, x_n)$ which is defined in (1) and $n_i = \deg f_i$ for $i = 1, \ldots, n$. Then

$$D(u/v) = \prod_{i=1}^{n} d(w_i/v)$$

Also the equality $D(u/v) = \prod_{i=1}^{n} e_i n_i$ holds.

**Proof:** Using the Theorem. 2.1. of [2] it can be seen that

$$d(u/v) = [K(x_1, \ldots, x_n) : K(r_1, \ldots, r_n)] = \prod_{i=1}^{n} \deg r_i = \prod_{i=1}^{n} d(w_i/v).$$

Since

$$d(w_i/v) = n_i f(w_i/v)$$

for $i = 1, \ldots, n$ the equality $D(u/v) = \prod_{i=1}^{n} e_i n_i$ holds.

**Theorem 2.4:** Let $u$ be the r.t. extension of $v$ to $K(x_1, \ldots, x_n)$ which is defined in (1). Then the inequality

$$D(u/v) \geq \prod_{i=1}^{n} e(w_i/v)f(w_i/v)$$

holds.

**Proof:** According to the Theo.2.2. and Corollary 2.2. of [2]

$$d(w_i/v) \geq e(w_i/v)f(w_i/v)$$

for $i = 1, \ldots, n$ then using the Theo. 2.3 the proof is completed.
Theorem 2.5: The equality $D(u/v) = \prod_{i=1}^{n} e(w_i/v)f(w_i/v)$ holds if one of the following conditions is satisfied.

i) $rank_v = 1$ and $chark_v = 0$

ii) $rank_v = 1$ and $v$ is discrete

iii) $v$ is Henselian and $chark_v = 0$.

Proof: According to Corollary 2.5 and Corollary 2.6. of [1] if one of this conditions is satisfied the equality $d(w_i/v) = e(w_i/v)f(w_i/v)$ holds for $i = 1, ..., n$. Therefore using the Theorem 2.3 the proof is obtained.

Corollary 2.6: Let $E$ and $F$ be the ramification index and the residue degree of $u/v$ respectively. Then the inequality

$$D(u/v) \geq EF$$

holds.

Proof: Since the value group of $u$ is $G_u = G_{v_{a_1},...,a_n} + \mathbb{Z}\gamma_1 + ... + \mathbb{Z}\gamma_n$

$$E = e(u/v) = \bar{e}[G_{v_{a_1},...,a_n} : G_v]$$

where $\bar{e}$ is the smallest multiple of $e_i$ for $i = 1, ..., n$. Therefore using the Theo. 2.3. it is obtained that

$$D(u/v) \geq \prod_{i=1}^{n} e(w_i/v)f(w_i/v) = \prod_{i=1}^{n} e_i[G_{v_{a_i}} : G_v] [k_{v_{a_i}} : k_v] \geq EF.$$ 

References


[5] F. Öke, H. İşcan, An introduction to extension of valuations on $K$ to $K(x,y)$, 


Received: December, 2008