On Absolute Valued Algebras Satisfying

\[(x, x, x) = 0, (x^2, x, x) = 0, (x, x, x^2) = 0, (x^2, x^2, x) = 0 \text{ or } (x, x^2, x^2) = 0\]

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Abstract

Let \( A \) be an absolute valued algebra containing a nonzero central element \( a \), we give some conditions imply that the norm of \( A \) comes from an inner product, this generalizes previously known results in [7], [8] and [3]. Moreover \( A \) is finite dimensional in the following cases:

1. \( A \) satisfies \((x^2, x, x) = 0 \) or \((x, x, x^2) = 0\),
2. \( A \) satisfies \((x^2, x^2, x) = 0 \) or \((x, x^2, x^2) = 0 \) with \((a|a^2) = 0\),
3. \( A \) satisfies \((x, x, x) = 0\).

In the first cases \( A \) is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \), some variants on the original proof are introduced [2, Theorem 3.6], in the second case \( A \) is isomorphic to \( \mathbb{C} \) and \( A \) is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}^*, \mathbb{O} \) or \( \mathbb{O}^* \) in the last case. This is more general than those results in [5], [6] and [7]. It may be conjectured that every absolute valued algebra containing a nonzero central element is an inner product space.

Keywords: Absolute valued algebras, flexible algebras, central element, central idempotent, inner product

1 Introduction

Absolute valued algebras are those real or complex algebras not necessarily associative or finite dimensional. It is shown that any absolute valued algebra containing a central idempotent must be an algebra with an involution [7], reciprocally is true for any finite dimensional absolute valued algebra with
an involution must contain a central idempotent [6]. As main results Rochdi [4] proved that any absolute valued algebra with an involution and satisfying

\((x^i, x^j, x^k) = 0\) for fixed \(i,j,k \in \{1,2\}\) (where \((\ldots, \ldots)\) means associator), is finite dimensional and he gave a description for such algebras. It is natural to study those absolute valued algebras by replacing the original assumption central idempotent by a weaker one central element, we prove that, if \(A\) is an absolute valued real algebra containing a central element \(a\) and satisfying one of the following conditions.

1. \(x^2a^2 = x^2a^2\) such that \((a|x) = 0,\)
2. \(x^2 = -\|x\|^2a^2\) such that \((a|x) = 0,\)
3. There exists \(b \in A\) such that \(\|b\| = 1\) and \(a = a^2b\)
4. \((a^2, a^2, a) = 0\) and \((a|a^2) = 0\)
5. \((x, x, x) = 0\) for all \(x \in A\)

Then \(A\) is an inner product space (Proposition 3.1, Lemmas 3.2, 3.4 and 3.5), this result is an important generalization of a results given in [7], [8] and [3]. Moreover if \(A\) satisfying \((x^2, x, x) = 0\) or \((x, x^2, x) = 0\), then \(A\) is finite dimensional (Theorems 3.8, 3.12 and 3.13) and we have the two following cases.

1. If \((a|a^2) = 0\), then \(A\) is isomorphic to \(\mathbb{C}\),
2. If \((a|a^2) \neq 0\), then \(a\) is a unit element of \(A\) and \(A\) is isomorphic to \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{O}\).

And if \(A\) satisfying \((x^2, x^2, x) = 0\) or \((x, x^2, x^2) = 0\) with \((a|a^2) = 0\), then \(A\) is isomorphic to \(\mathbb{C}\) (Theorems 3.15 and 3.16). Also we show that if \(A\) satisfying \((x, x, x) = 0\) then \(A\) is finite dimensional in the two following cases:

1. If \((a|a^2) = 0\), then \(A\) is isomorphic to \(\mathbb{C}\) or \(\mathbb{C}^*\) (Theorem 3.17),
2. If \((a|a^2) \neq 0\), then \(A\) is flexible and is isomorphic to \(\mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}^*, \mathbb{O}\) or \(\mathbb{O}^*\) (Theorem 3.18).

This latter is more general than those results in [5], [6] and [7].
## 2 Preliminary Notes

Throughout the paper, the word algebra refers to a non-necessarily associative algebra over the field of real numbers $\mathbb{R}$.

**Definitions 2.1.** Let $B$ be an arbitrary algebra.

i) We say that $B$ is algebraic, if for every $x$ in $B$, the subalgebra $B(x)$ of $B$ generated by $x$ is finite dimensional \([1]\).

ii) We mean by a nonzero central element in $B$, a nonzero element which commute with all elements of the algebra $B$.

iii) $B$ is termed normed (resp, absolute valued) if it is endowed with a space norm: $\|\cdot\|$ such that $\|xy\| \leq \|x\|\|y\|$ (resp, $\|xy\| = \|x\|\|y\|$), for all $x, y \in B$.

iv) $B$ is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product $(\cdot, \cdot)$ such that $(\cdot, \cdot) : B \times B \to \mathbb{R}$ $(x, y) \mapsto (x|y) = 4^{-1}(\|x+y\|^2 - \|x-y\|^2)$.

The most natural examples of absolute valued algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (the algebra of Hamilton quaternion), $\mathbb{O}$ (the algebra of Cayley numbers), $\mathbb{C}^*$ (the algebra of Mc Clay), $\mathbb{H}^*$ (parra-quaternion algebra), $\mathbb{O}^*$ (parra-octonion algebra) and $\mathbb{P}$ (pseudo-octonion) \([11]\), with norms equal to their usual absolute values \([10]\) and \([13]\). The algebra $\mathbb{C}^*$ (resp, $\mathbb{H}^*$ and $\mathbb{O}^*$) obtained by replacing the product of $\mathbb{C}$ (resp, $\mathbb{H}$ and $\mathbb{O}$) with the one defined by $x \circ y = x^*y^*$, where $*$ means the standard involution of $B$.

We need the following results:

**Lemma 2.2.** \([15]\) If all the elements of a subset $B$ of any absolute valued algebra $A$ commute with each other, then the linear hull spanned by $B$ is pre-Hilbert space.

**Theorem 2.3.** \([15]\) If $A$ is an absolute valued algebra with a unit element then $A$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

**Theorem 2.4.** \([15]\) A commutative absolute valued algebra is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{C}^*$.

**Theorem 2.5.** \([5]\) Let $A$ be a pre-Hilbert absolute valued algebra satisfying $(x, x, x) = 0$ for all $x \in A$. Then $A$ is finite dimensional and is isotope to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. 
Theorem 2.6. [7] Any finite dimensional absolute valued algebra satisfying 
\((x, x, x) = 0\) for all \(x \in A\) is isomorphic to \(\mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{O}, \mathbb{O}^*\) or \(\mathbb{P}\).

Corollary 2.7. [7] The pseudo-octonion algebra \(\mathbb{P}\) is the only (up to an isomorphism) finite dimensional absolute valued algebra which satisfies the identity \((x, x, x) = 0\) and does not contains a central idempotent.

Theorem 2.8. [8] If \(A\) is an absolute valued algebra containing a nonzero central element \(a\) which satisfies \((a, a, a^2) = 0\) then \(A\) is an inner product space.

Theorem 2.9. [12] The norm of any absolute valued algebra \(A\) with left unit \(e\) comes from an inner product \((., .)\) satisfying \((ab|c) = -(b|ac)\) and \(a(ab) = -\|a\|^2b\) for all \(a, b, c \in A\) with \(a\) orthogonal to \(e\).

Theorem 2.10. [4] Any absolute valued algebra \(A\) with involution satisfying \((x^i, x^j, x^k) = 0\) for fixed \(i, j, k \in \{1, 2\}\) is finite dimensional. The following table specifies the isomorphisms classes:

<table>
<thead>
<tr>
<th>(A) satisfies ((., ., .) = 0)</th>
<th>The list of isomorphisms classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, x) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{C}^<em>, \mathbb{H}, \mathbb{O}, \mathbb{O}^</em>)</td>
</tr>
<tr>
<td>((x, x, x^2) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})</td>
</tr>
<tr>
<td>((x, x^2, x) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H}, \mathbb{O}, \mathbb{O}^*)</td>
</tr>
<tr>
<td>((x, x^2, x^2) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})</td>
</tr>
<tr>
<td>((x^2, x) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})</td>
</tr>
<tr>
<td>((x^2, x^2) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H}, \mathbb{O}, \mathbb{O}^*)</td>
</tr>
<tr>
<td>((x^2, x^2, x) = 0)</td>
<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})</td>
</tr>
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<td>(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H}, \mathbb{O}, \mathbb{O}^*)</td>
</tr>
</tbody>
</table>

3 Absolute Valued Algebras Satisfying \((x^2, x, x) = 0, (x, x, x^2) = 0, (x^2, x^2, x) = 0, (x, x^2, x^2) = 0\) or \((x, x, x) = 0\)

Throughout this paper, we suppose that \(\|a\| = 1\)

3.1 Some Conditions On Absolute Valued Algebras

we give some conditions imply that \(A\) is an inner product space.

Proposition 3.1. Let \(A\) be an absolute valued algebra containing a central element \(a\) and let \(x\) be a element in \(A\). If \(x\) is orthogonal to \(a\) in the inner product space \([a, x]\), then the following are equivalent:
1. $x^2a^2 = x^2a^2$.
2. $x^2 = -\|x\|^2a^2$.
3. $A$ is an inner product space.

**Proof.** 1) $\Rightarrow$ 2) Assuming that $\|x\| = 1$, we have
\[\|x^2 - a^2\| = \|x - a\|\|x + a\| = 2\]
According to Lemma 2.2 we get $x^2 = -a^2$.

2) $\Rightarrow$ 1) is clear

2) $\Rightarrow$ 3) let $u = \alpha a + \beta x$ and $v = \gamma a + \delta y$ be norm-one elements in $A$ where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $x, y \in \{a\}^\perp$ such that $\|x\| = \|y\| = 1$. According to Schoenberg's Theorem [14], it is sufficient to show that the inequality $\|u + v\|^2 + \|u - v\|^2 \geq 4$ holds. Using Lemma 2.2 and since $x^2 = y^2 = -a^2$, then
\[\|u + v\|^2 + \|u - v\|^2 = (\|\alpha + \gamma\|a + (\beta x + \delta y))^2 + (\|\alpha - \gamma\|a + (\beta x - \delta y))^2 = (\alpha + \gamma)^2 + \|\beta x + \delta y\|^2 + (\alpha - \gamma)^2 + \|\beta x - \delta y\|^2 = (\alpha + \gamma)^2 + (\alpha - \gamma)^2 + \|\beta x + \delta y\|^2 + \|\beta x - \delta y\|^2 = \geq 2\alpha^2 + 2\gamma^2 + \|2(\beta^2 + \delta^2)a^2\| = 2\alpha^2 + 2\gamma^2 + 2\beta^2 + 2\delta^2 = 4\]
This implies that $A$ is an inner product space.

3) $\Rightarrow$ 2) Assuming that $\|x\| = 1$, we have $\|(a + x)^2\| = 4$ and since
\[(a^2|ax) = (x^2|ax) = (a|x) = 0,\]
then $(a^2|x^2) = -1$. Moreover
\[\|a^2 + x^2\|^2 = \|a^2\|^2 + 2(a^2|x^2) + \|x^2\|^2 = 1 - 2 + 1 = 0 \quad (1)\]
So (1) gives $x^2 = -a^2$, and then $x^2 = -\|x\|^2a^2$ for all $x \in \{a\}^\perp$.

**Lemma 3.2.** Let $A$ be an absolute valued algebra containing a central element $a$. If there exists $b \in A$ such $\|b\| = 1$ and $a = a^2b$, then $A$ is an inner product space.

**Proof.** Let $x$ be a norm one in $A$ and suppose that $(a|x) = 0$ in the inner product space $[a, x]$, then we have
\[2 = \|x - a\|\|x + a\| = \|x^2 - a^2\| = \|x^2b - a^2b\| = \|x^2b - a\|.
As $(x^2)b = a(x^2b)$, then by Lemma 2.2 we get $x^2b = -a = -a^2b$. Hence the result is concluded by a simplification by $b$ and using Proposition 3.1.
Corollary 3.3. Let $A$ be an absolute valued algebra containing a central algebraic element $a$, then $A$ is an inner product space.

Proof. As $A(a)$ is finite dimensional, then $A(a)$ is a division algebra. Therefore the operator $L_{a^2}$ of left multiplication by $a^2$ on $A(a)$ is bijective, and there exists $b \in A(a)$ such $\|b\| = 1$ and $a = L_{a^2}(b) = a^2b$. Then the result is consequence of the lemma 3.2.

Lemma 3.4. Let $A$ be an absolute valued algebra containing a central element $a$ which satisfies $(a^2, a^2, a) = 0$ and is orthogonal to $a^2$ in the inner product space $\langle a, a^2 \rangle$. Then $A$ is an inner product space.

Proof. By Lemma 3.2, it is sufficient to prove that there exists $b \in A$ such $\|b\| = 1$ and $a = a^2b$. We have

$$\|a - a^2a\| = \|a^2\|\|a-aa^2\| = \|a^2a - a^2(a^2a)\| = \|a^2a - (a^2)^2a\| = \|a - a^2\||a + a^2|.$$

By Lemma 2.2 the subspace $\langle a, a^2 \rangle$ is a Hilbert space. As $(a|a^2) = 0$, then $\|a - a^2\||a + a^2| = 2$, that is, $\|a - a^2a\| = 2$. According to Lemma 2.2 we get $aa^2 = -a$. Hence $A$ is an inner product space.

Lemma 3.5. Let $A$ be an absolute valued algebra containing a central element $a$ and satisfying $(x, x, x) = 0$ for all $x \in A$, then $xa^2 = a^2x$ and $A$ is an inner product space.

Proof. Let $x$ be a element in $A$, we have $(x + a, x + a, x + a) = 0$. Then

$$0 = (x + a, x + a, x + a) = (x, x, a) + (x, a, x) + (x, a, a) + (a, x, x) + (a, a, x).$$

Replacing $x$ by $-x$, we get

$$(x, x, a) + (x, a, x) - (x, a, a) + (a, x, x) - (a, a, x) = 0$$

Adding these two equalities, we have

$$0 = (x, a, a) + (a, a, x) = xa^2 - a^2x,$$

Then $xa^2 = a^2x$ for all $x \in A$, by Proposition 3.1 we conclude that $A$ is an inner product space.

Lemma 3.6. Let $A$ be an absolute valued algebra whose norm comes from an inner product and contains a nonzero central element $a$. Then $xy + yx = -2(x|y)a^2$ for all $x, y \in \{a\}^\perp$.

Proof. We may assume $\|x\| = \|y\| = 1$. We have $x, y \in \{a\}^\perp$ then $(x + y)^2 = -\|x + y\|^2a^2$ (Proposition 3.1), hence $xy + yx = -2(x|y)a^2$. 

3.2 Absolute Valued Algebras Satisfying \((x, x, x^2) = 0\) or \((x^2, x, x) = 0\)

In this section we show that if \(A\) is an absolute valued algebra containing a central element \(a\) and satisfying \((x, x, x^2) = 0\) or \((x^2, x, x) = 0\). Then \(A\) is finite dimensional and is isomorphic to \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{O}\). This generalizes known results in [9]

3.2.1 Case \((a|a^2) = 0\)

**Lemma 3.7.** Let \(A\) be an absolute valued algebra containing a central element \(a\) and satisfying \((x^2, x, x) = 0\) for all \(x \in A\). If \((a|a^2) = 0\), then \(e = -a^2\) is a left unit of \(A\).

**Proof.** \(A\) is an inner product space (Theorem 2.8), let \(b \in \{a\}^\perp\) such that \(\|b\| = 1\). We have \(b^2 = -a^2 = e\) (Proposition 3.1), then

\[
b^2 = e = (b^2)^2 = (b^2)b = (eb)\]

This implies \(eb = b\) and similarly, we get \(ea = ae = a\). Thus \(ey = y\) for all \(y \in A\).

**Theorem 3.8.** Let \(A\) be an absolute valued algebra containing a central element \(a\) and satisfying \((x^2, x, x) = 0\) for all \(x \in A\). If \((a|a^2) = 0\), then \(A\) is isomorphic to \(\mathbb{C}\).

**Proof.** According to Theorem 2.8, we have \(A\) is an inner product space. Since \((a|a^2) = 0\), then \(A(a, a^2)\) is a commutative subalgebra of \(A\) (Lemma 3.7) with unit \(e = -a^2\), hence \(A(a, a^2)\) and is isomorphic to \(\mathbb{C}\) (Theorem 2.3). \(e = -a^2\) is a left unit of \(A\) (Lemma 3.7), and let \(b \in \{e, a\}^\perp\) such that \(\|b\| = 1\). We have \(b^2 = -a^2 = e\) (Proposition 3.1) and \(eb = b\). Moreover

\[
(ab|e) = (ab|b^2) = (a|b) = 0 \quad \text{and} \quad (ab|b) = (ab|eb) = (a|e) = 0
\]

Then

\[
(ab)b = -b(ab) \quad \text{(Lemma 3.6)}
\]

\[
= -b(ba)
\]

\[
= -\|b\|^2 a \quad \text{(Theorem 2.9)}
\]

\[
= -a
\]

Since \(((a + b)^2, a + b, a + b) = 0\) and \(b^2 = -a^2\), then

\[
[ab(a + b)](a + b) = ab(a + b)^2
\]

\[
= 2(ab)^2
\]

\[
= 2e
\]
Consequently

\[
2e = [ab(a + b)](a + b) \\
= [(ab)a + (ab)b](a + b) \\
= [a(ab) - b(ab)](a + b) \quad \text{(Lemma 3.6)} \\
= (-b + a)(a + b) \quad \text{(Theorem 2.9)} \\
= -2e
\]

Which is absurd, hence \( A = A(e, a) \) is isomorphic to \( \mathbb{C} \).

3.2.2 Case \((a|a^2) = m \neq 0\)

**Theorem 3.9.** Let \( A \) be an absolute valued algebra containing a central element \( a \) and satisfying \((x^2, x, x) = 0\) for all \( x \in A \). Let \( d = a^2 - ma \) such that \((d|a) = 0\), then \( A(a, d) \) is isomorphic to \( \mathbb{C} \).

**Proof.** By Theorem 2.8, \( A \) is an inner product space. Assume that \((a|a^2) = m \neq 0\) and let \( d = a^2 - ma \). We have \((d|a) = 0\), then

\[
d^2 = -\|d\|^2 a^2 = -(1 - m^2)a^2.
\]

That is

\[
-(1 - m^2)a^2 = (a^2 - ma)^2 \\
= (a^2)^2 - 2maa^2 + m^2a^2 \\
= (a^2a)a - 2ma^2a + m^2a^2 \\
= (a^2a - 2ma^2 + m^2a)a
\]

This gives

\[
-(1 - m^2)a = a^2a - 2ma^2 + m^2a \\
-(1 - m^2)a = (a^2 - ma)a - m(a^2 - ma) \\
-(1 - m^2)a = da - md
\]

Hence \( ad = da = -(1 - m^2)a + md \), this implies that \( A(a, d) \) is a two-dimensional subalgebra of \( A \) and is isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^\ast \) (Theorem 2.4). If \( A(a, d) \) is isomorphic to \( \mathbb{C}^\ast \), there exist a nonzero two idempotents \( e_1 \) and \( e_2 \) such that \( e_1e_2 + e_1 + e_2 = 0 \). We have

\[
(e_1 + 2e_2)^2 = e_1 + 4e_2 + 4e_1e_2 = -3e_1
\]
Moreover
\[
0 = ((e_1 + 2e_2)^2, e_1 + 2e_2, e_1 + 2e_2)
= (-3e_1, e_1 + 2e_2, e_1 + 2e_2)
= -3[e_1(e_1 + 2e_2)](e_1 + 2e_2) + 3e_1(e_1 + 2e_2)^2
= -3(e_1 + 2e_1e_2)(e_1 + 2e_2) - 9e_1
= 3(e_1 + 2e_2)(e_1 + 2e_2) - 9e_1
= -9e_1 - 9e_1.
\]

Which is absurd. Then $A(a, d)$ is isomorphic to $\mathbb{C}$.

From the last result we conclude there exists a nonzero idempotent $e \in A$ and a nonzero element $i \in A$ such that $ie = ei$, $\|i\| = 1$ and $i^2 = -e$. We put $a = \alpha e + \beta i$ with $\alpha, \beta \in \mathbb{R}$ and we can choose $d = ia$. Then we get

**Lemma 3.10.** Let $A$ be an absolute valued algebra containing a central element $a = \alpha e + \beta i$ (notation above) and satisfying $(x^2, x, x) = 0$ for all $x \in A$. Then $a^2u = ue$ for all $u \in \{e, i\}^\perp$.

**Proof.** By Theorem 2.8, $A$ is an inner product space. Let $u \in \{e, i\}^\perp$ be a norm one element and $d = ia = -\beta e + \alpha i$ (notation above), we have $(d|u) = 0$ then $du = -ud$ (Lemma 3.6). Since $au = ua$, we obtain

$$
\alpha eu + \beta iu = \alpha ue + \beta ui \quad \text{and} \quad -\beta eu + \alpha iu = \beta ue - \alpha ui
$$

From these equalities, we get
\[
eu = \alpha^2 ue + \alpha \beta ui - \beta^2 ue + \alpha \beta ui
= (\alpha^2 - \beta^2)ue + 2\alpha \beta ui
= ua^2
\]

**Lemma 3.11.** Let $A$ be an absolute valued algebra containing a central element $a$ and satisfying $(x^2, x, x) = 0$ for all $x \in A$. Then:

1. $[(ay)a]y = [(ay)y]a = (ay)^2$.
2. $[(ay)y]y + \|y\|^2[(ay)a]a = 2(a|y)(ay)^2$

for all $y \in A$.

**Proof.** Since $A$ is an inner product (Theorem 2.8) and $y - (a|y)a$ is orthogonal to $a$, then
\[
(y - (a|y)a)^2 = -\|y - (a|y)a\|^2 a^2 = -(\|y\|^2 - (a|y)^2) a^2
\]
This gives $y^2 + \|y\|^2a^2 = 2(a|y)a$, we can assume that $(a|y)^2 \neq 1$ and (without loss of generality) $\|y\| = 1$. We have

$$[(a + y)^2] = (a^2 + y^2 + 2ay)^2$$

$$((a + y)^2)(a + y) = (2(a|y)a + 2ay)^2$$

$$((2(a|y)a + 2ay)(a + y))(a + y) = 4(1 + (a|y))^2(ay)^2$$

$$2(1 + (a|y))((ay)(a + y))(a + y) = 4(1 + (a|y))((ay)^2)$$

$$((ay)a + ((ay)a)y + ((ay)y)a + ((ay)y)y) = 2(1 + (a|y))((ay)^2)$$

We replace $y$ by $-y$, we get

$$-[(ay)a]a + [(ay)a]y + [(ay)y]a - [(ay)y]y = 2(1 - (a|y))(ay)^2$$

Adding the last two equalities, we conclude that $[(ay)a]y + [(ay)y]a = 2(ay)^2$ and $[(ay)y]y + [(ay)a]a = 2(a|y)(ay)^2$. Since $\|[(ay)a]y + [(ay)y]a\|^2 = \|2(ay)^2\|^2 = 4$, then $[(ay)a]y$|[(ay)y]a = 1 . Hence $[(ay)a]y = [(ay)y]a = (ay)^2$, this completes the proof.

**Theorem 3.12.** Let $A$ be an absolute valued algebra containing a central element $a$ and satisfying $(x^2, x, x) = 0$ for all $x \in A$. Then $A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

**Proof.** It suffices to show that $a$ is a unit element of $A$. Since $A$ is an inner product space (Theorem 2.8) and let $u \in \{e, i\}^+$ be a norm one element, we have

$$((au + a)a)(u + e) = (au + a)^2 \quad (\text{Lemma 3.11})$$

$$((au + a)a)(u + e) = (au + a)^2$$

$$((au)a)u + ((au)a)e + a^2u + a^2 = (au)^2 + 2(au)a + a^2$$

$$((au)a)e + a^2u = 2(au)a$$

Since $\|((au)a)e + a^2u\|^2 = \|2(au)a\|^2 = 4$, then $((au)a)e|a^2u) = 1$. Hence $((au)a)e = a^2u = (au)a$, this gives $((au)a)e = a^2u = ue$ (Lemma 3.10) or $(au)a = u$, that is, $u = (au)a = a^2u = ue$. On the other hand, we have $u^2 = -a^2$ then

$$(a^2)^2 = (u^2)^2$$

$$= (u^2u)u$$

$$= -(a^2u)u$$

$$= -(ue)u \quad (\text{Lemma 3.10})$$

$$= -u^2$$

$$= a^2$$
This implies $a^2 = a$, and therefore $a = e$ or $a = -e$. Thus $a$ is a unit element of $A$. Then $A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

Similarly, we can get all preceding results if $A$ satisfying $(x, x, x^2) = 0$

**Theorem 3.13.** Let $A$ be an absolute valued algebra containing a central element $a$ and satisfying $(x, x, x^2) = 0$ for all $x \in A$. Then $A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (Theorem 2.3).

### 3.3 Absolute Valued Algebras Satisfying $(x, x^2, x^2) = 0$ or $(x^2, x^2, x) = 0$

In this section we prove that if $A$ is an absolute valued algebra containing a central element $a$ and satisfying $(x, x^2, x^2) = 0$ or $(x^2, x^2, x) = 0$ with $(a|a^2) = 0$. Then $A$ is finite dimensional and is isomorphic to $\mathbb{C}$.

**Lemma 3.14.** Let $A$ be an absolute valued algebra containing a central element $a$ and satisfying $(x^2, x^2, x) = 0$ for all $x \in A$. If $(a|a^2) = 0$, then $e = -a^2$ is a left unit of $A$.

**Proof.** $A$ is an inner product space (Lemma 3.4), let $b \in \{a\}^\perp$ such that $\|b\| = 1$. We have $b^2 = -a^2 = e$ (Proposition 3.1), then

$$b^2b = eb = (b^2)^2b = b^2(b^2b) = e(eb)$$

This implies $eb = b$ and similarly, we get $ea = ae = a$. Thus $ey = y$ for all $y \in A$.

**Theorem 3.15.** Let $A$ be an absolute valued algebra containing a central element $a$ and satisfying $(x^2, x^2, x) = 0$ for all $x \in A$. If $(a^2|a) = 0$, then $A$ is isomorphic to $\mathbb{C}$.

**Proof.** According to Lemma 3.4, we have $A$ is an inner product space and $aa^2 = a^2a = -a$. Then $A(a, a^2)$ is a commutative subalgebra of $A$ with unit element $e = -a^2$, hence $A(a, a^2)$ is isomorphic to $\mathbb{C}$ (Theorem 2.3). $e = -a^2$ is a left unit of $A$ (Lemma 3.14), let $b \in \{e, a\}^\perp$ such that $\|b\| = 1$. We have $b^2 = -a^2 = e$ (Proposition 3.1) and $eb = b$. Moreover

$$(ab|a) = (ab|ae) = (b|e) = 0 \text{ and } (ab|e) = -(ab|a^2) = -(a|b) = 0$$

Since $((a + b)^2, (a + b)^2, a + b) = 0$ and $b^2 = -a^2$, then

$$(ab)^2(a + b) = (ab)[(ab)(a + b)]$$

$$e(a + b) = (ab)[(ab)a + (ab)b]$$

$$a + b = (ab)[(ab)a] + (ab)[(ab)b]$$

$$a + b = -\|ab\|^2a - \|(ab)b\|^2 (Theorem 2.9)$$

$$a + b = -a - b$$
Which is absurd, hence \( A = A(e, a) \) is isomorphic to \( \mathbb{C} \).

Similarly we get the following result

**Theorem 3.16.** Let \( A \) be an absolute valued algebra containing a central element \( a \) and satisfying \((x, x^2, x^2) = 0 \) for all \( x \in A \). If \((a^2|a) = 0 \), then \( A \) is isomorphic to \( \mathbb{C} \).

### 3.4 Absolute Valued Algebras Satisfying \((x, x, x) = 0\)

In this section we prove that if \( A \) is an absolute valued algebra containing a central element \( a \) and satisfying \((x, x, x) = 0 \). Then \( A \) is flexible and is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}^*, \mathbb{O} \) or \( \mathbb{O}^* \).

**Theorem 3.17.** Let \( A \) be an absolute valued algebra containing a central element \( a \) and satisfying \((x, x, x) = 0 \) for all \( x \in A \). If \((a|a^2) = 0 \), then \( A \) is finite dimensional and is isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^* \).

**Proof.** The first assertion is consequence of Lemma 3.5 and Theorem 2.5. Since \((a|a^2) = 0 \) and \( A \) is an inner product space, then by Proposition 3.1 we get \((a^2) = -a^2 \). Hence \( e = -a^2 \) is a central idempotent of \( A \), using Theorem 2.6 and Corollary 2.7 we get \( A \) is flexible and is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}^*, \mathbb{O} \) or \( \mathbb{O}^* \). Let \( x \in A \) be a orthonormal element to \( a \) and \( a^2 \), we have \( x^2 = -a^2 = e \) and \( a^2x = xa^2 \) (Lemma 3.5 and Proposition 3.1). Then

\[
0 = x^2 - e = x^2 - (a^2)^2 = (x - a^2)(x + a^2)
\]

This implies \( x = a^2 \) or \( x = -a^2 \), which is absurd. Then \( A \) is a two dimensional commutative algebra and therefore \( A \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^* \).

**Theorem 3.18.** Let \( A \) be an absolute valued algebra containing a central element \( a \) and satisfying \((x, x, x) = 0 \) for all \( x \in A \). Then \( a \) is a central idempotent of \( A \) and \( A \) is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}^*, \mathbb{O} \) or \( \mathbb{O}^* \).

**Proof.** The Lemma 3.5, Theorem 2.6 and Corollary 2.7 imply that \( A \) is flexible and is finite dimensional. Let \( x \in \{a^2, a\}^\perp \) such that \( \|x\| = 1 \), we have \( xa^2 = a^2x \) (Lemma 3.5) and \( ax = xa \), then \(-a^2 = x^2 = -a^2\) (Proposition 3.1). This means that \( a^2 = a \) or \( a^2 = -a \), and therefore \( a \) is a central idempotent of \( A \) and \( A \) is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}^*, \mathbb{O} \) or \( \mathbb{O}^* \).

Now, we can generalize some results in the Theorem 2.9
Theorem 3.19. Any absolute valued algebra $A$ containing a nonzero central element $a$ and satisfying $(x^2, x, x) = 0$, $(x, x, x^2) = 0$ or $(x, x, x) = 0$ \[\text{resp}\] $(x^2, x^2, x) = 0$ or $(x, x^2, x^2) = 0$ with $(a|a^2) = 0$ is finite dimensional. The following table specifies the isomorphisms classes:

<table>
<thead>
<tr>
<th>$(x, x, x)$ satisfies</th>
<th>The list of isomorphisms classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, x, x) = 0$</td>
<td>$\mathbb{R}, \mathbb{C}, \mathbb{C}^<em>, \mathbb{H}, \mathbb{H}^</em>, \mathbb{O}, \mathbb{O}^*$</td>
</tr>
<tr>
<td>$(x, x, x^2) = 0$</td>
<td>$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$</td>
</tr>
<tr>
<td>$(x^2, x, x) = 0$</td>
<td>$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$</td>
</tr>
<tr>
<td>$(x^2, x^2, x) = 0$ with $(a</td>
<td>a^2) = 0$</td>
</tr>
<tr>
<td>$(x, x^2, x^2) = 0$ with $(a</td>
<td>a^2) = 0$</td>
</tr>
</tbody>
</table>

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References


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