A Note on $\sigma$-Reversibility and $\sigma$-Symmetry of Skew Power Series Rings

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Abstract

Let $R$ be a ring and $\sigma$ an endomorphism of $R$. In this note, we study the transfer of the symmetry ($\sigma$-symmetry) and reversibility ($\sigma$-reversibility) from $R$ to its skew power series ring $R[[x;\sigma]]$. Moreover, we study on the relationship between the Baerness, quasi-Baerness and p.p.-property of a ring $R$ and these of the skew power series ring $R[[x;\sigma]]$ in case $R$ is right $\sigma$-reversible. As a consequence we obtain a generalization of [10].

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1 Introduction

Throughout this paper $R$ denotes an associative ring with identity and $\sigma$ denotes a nonzero non identity endomorphism of a given ring.

Recall that a ring is reduced if it has no nonzero nilpotent elements. Lambek [16], called a ring $R$ symmetric if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. Every reduced ring is symmetric ([19, Lemma 1.1]) but the converse does not hold by [1, Example II.5]. Cohen [8], called a ring $R$ reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. It is obvious that commutative rings are symmetric and symmetric rings are reversible, but the converse does not hold by [1, Examples I.5 and II.5] and [17, Examples 5 and 7]. From [3], a ring $R$ is called right (left) $\sigma$-reversible if whenever $ab = 0$ for $a, b \in R$, $b\sigma(a) = 0$ ($\sigma(b)a = 0$). $R$ is called $\sigma$-reversible if it is both right and left $\sigma$-reversible. Also, by [15], a ring $R$ is called right (left) $\sigma$-symmetric if whenever $abc = 0$ for $a, b, c \in R$, $a\sigma(c)b = 0$
(σ(b)ac = 0). R is called σ-symmetric if it is both right and left σ-symmetric. Clearly right σ-symmetric rings are right σ-reversible.

Rege and Chhawchharia [18], called a ring R an Armendariz if whenever polynomials \( f = \sum_{i=0}^{n} a_i x^i, \ g = \sum_{j=0}^{m} b_j x^j \in R[x] \) satisfy \( fg = 0 \), then \( a_i b_j = 0 \) for each \( i, j \). The Armendariz property of a ring was extended to one of skew polynomial ring in [11]. For an endomorphism \( \sigma \) of a ring \( R \), a skew polynomial ring (also called an Ore extension of endomorphism type) \( R[x; \sigma] \) of \( R \) is the ring obtained by giving the polynomial ring over \( R \) with the new multiplication \( xr = \sigma(x)r \) for all \( r \in R \). Also, a skew power series ring \( R[[x; \sigma]] \) is the ring consisting of all power series of the form \( \sum_{i=0}^{\infty} a_i x^i \) \( (a_i \in R) \), which are multiplied using the distributive law and the Ore commutation rule \( xa = \sigma(a)x \) for all \( a \in R \). According to Hong et al. [11], a ring \( R \) is called σ-skew Armendariz if whenever polynomials \( f = \sum_{i=0}^{n} a_i x^i \) and \( g = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma] \) satisfy \( fg = 0 \) then \( a_i \sigma^j(b_j) = 0 \) for each \( i, j \). Baser et al. [4], introduced the concept of \( \sigma \)-(sps) Armendariz rings. A ring \( R \) is called \( \sigma \)-(sps) Armendariz if whenever \( pq = 0 \) for \( p = \sum_{i=0}^{\infty} a_i x^i, \ q = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \sigma]] \), then \( a_i b_j = 0 \) for all \( i \) and \( j \). According to Krempa [14], an endomorphism \( \sigma \) of a ring \( R \) is called rigid if \( a \sigma(a) = 0 \) implies \( a = 0 \) for all \( a \in R \). We call a ring \( R \) \( \sigma \)-rigid if there exists a rigid endomorphism \( \sigma \) of \( R \). Note that any rigid endomorphism of a ring \( R \) is a monomorphism and \( \sigma \)-rigid rings are reduced by Hong et al. [10]. Also, by [15, Theorem 2.8(1)], a ring \( R \) is \( \sigma \)-rigid if and only if \( R \) is semiprime right \( \sigma \)-symmetric and \( \sigma \) is a monomorphism, so right \( \sigma \)-symmetric (\( \sigma \)-reversible) rings are a generalization of \( \sigma \)-rigid rings.

In this note, we introduce the notion of \( \sigma \)-skew (sps) Armendariz rings which is a generalization of \( \sigma \)-(sps) Armendariz rings, and we study the transfert of the symmetry (\( \sigma \)-symmetry) and reversibility (\( \sigma \)-reversibility) from \( R \) to its skew power series ring \( R[[x; \sigma]] \). Also we show that \( R \) is \( \sigma \)-(sps) Armendariz if and only if \( R \) is \( \sigma \)-skew (sps) Armendariz and \( ab = 0 \) implies \( ab = 0 \) for \( a, b \in R \). Moreover, we study on the relationship between the Baerness, quasi-Baerness and p.p.-property of a ring \( R \) and these of the skew power series ring \( R[[x; \sigma]] \) in case \( R \) is right \( \sigma \)-reversible. As a consequence we obtain a generalization of [10].

2 \( \sigma \)-Reversibility and \( \sigma \)-Symmetry of Skew Power Series Rings

We introduce the next definition.

**Definition 2.1.** Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). A ring \( R \) is called \( \sigma \)-skew (sps) Armendariz if whenever \( pq = 0 \) for \( p = \sum_{i=0}^{\infty} a_i x^i, \ q = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \sigma]] \), then \( a_i \sigma^j(b_j) = 0 \) for all \( i \) and \( j \).
Every subring $S$ with $\sigma(S) \subseteq S$ of an $\sigma$-skew (sps) Armendariz ring is a $\sigma$-skew (sps) Armendariz ring. In the next, we give an example of a ring $R$ which is $\sigma$-skew (sps) Armendariz but not $\sigma$-(sps) Armendariz.

**Example 2.2.** Let $R$ be the polynomial ring $\mathbb{Z}_2[x]$ over $\mathbb{Z}_2$, and let the endomorphism $\sigma: R \rightarrow R$ be defined by $\sigma(f(x)) = f(0)$ for $f(x) \in \mathbb{Z}_2[x]$.

(i) $R$ is not $\sigma$-(sps) Armendariz because $\sigma$ is not a monomorphism.

(ii) $R$ is an $\sigma$-skew (sps) Armendariz ring (as in [11, Example 5]). Consider $R[[y;\sigma]] = \mathbb{Z}_2[x][[y;\sigma]]$. Let $p = \sum_{i=0}^{\infty} f_i y^i$ and $q = \sum_{j=0}^{\infty} g_j y^j \in R[[y;\sigma]]$. We have $pq = \sum_{\ell \geq 0} \sum_{i+j=\ell} f_i \sigma^i(g_j)y^\ell = 0$. If $pq = 0$ then $\sum_{\ell=0}^{\infty} f_i \sigma^i(g_j)y^\ell = 0$, for each $\ell \geq 0$. Suppose that there is $f_s \neq 0$ for some $s \geq 0$ and $f_0 = f_1 = \cdots = f_{s-1} = 0$, then $\sum_{i+j=s} f_i \sigma^i(g_j)y^{i+j} = 0 \Rightarrow f_s \sigma^s(g_0) = 0$, since $R$ is a domain then $g_0(0) = 0$. Also $\sum_{i+j=s+1} f_i \sigma^i(g_j)y^{i+j} = 0 \Rightarrow f_s \sigma^s(g_1) + f_{s+1} \sigma^{s+1}(g_0) = 0$, since $g_0(0) = 0$ then $f_s \sigma^s(g_1) = 0$ and so $g_1(0) = 0$ by the same method as above. Continuing this process, we have $g_j(0) = 0$ for all $j \geq 0$. Thus $f_i \sigma^i(g_j) = 0$ for all $i,j$.

We say that $R$ satisfies the condition $(C_\sigma)$, if whenever $a\sigma(b) = 0$ for $a, b \in R$, then $ab = 0$. By [4, Theorem 3.3(3iii)], if $R$ is $\sigma$-(sps) Armendariz then it satisfies $(C_\sigma)$ (so $\sigma$ is a monomorphism). If $R$ is an $\sigma$-skew (sps) Armendariz ring satisfying the condition $(C_\sigma)$ then $R$ is $\sigma$-(sps) Armendariz.

**Theorem 2.3.** A ring $R$ is $\sigma$-(sps) Armendariz ring if and only if it is $\sigma$-skew (sps) Armendariz and satisfies the condition $(C_\sigma)$.

**Proof.** ($\Leftarrow$). It is clear. ($\Rightarrow$). If $R$ is $\sigma$-(sps) Armendariz then it satisfies the condition $(C_\sigma)$. It suffices to show that if $R$ is $\sigma$-(sps) Armendariz then it is $\sigma$-skew (sps) Armendariz. The proof is similar as of [12, Theorem 1.8].

Let $p = \sum_{i=0}^{\infty} a_i x^i$ and $q = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\sigma]]$ with $pq = 0$. Note that $a_i b_j = 0$ for all $i$ and $j$. We claim that $a_i \sigma^i(b_j) = 0$ for all $i$ and $j$. We have $(a_0 + a_1 x + \cdots)(b_0 + b_1 x + \cdots) = 0$, then $a_0(b_0 + b_1 x + \cdots) + (a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + \cdots) = 0$. Since $a_0 b_j = 0$ for all $j$, we get

$$0 = (a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + \cdots)$$
$$0 = (a_1 + a_2 x + \cdots)x(b_0 + b_1 x + \cdots)$$
$$0 = (a_1 + a_2 x + \cdots)(\sigma(b_0)x + \sigma(b_1)x^2 + \cdots).$$

Put $p_1 = a_1 + a_2 x + \cdots$ and $q_1 = \sigma(b_0)x + \sigma(b_1)x^2 + \cdots$. Since $p_1 q_1 = 0$ then $a_i \sigma(b_j) = 0$ for all $i \geq 1$ and $j \geq 0$. We have, also

$$0 = a_1(\sigma(b_0)x + \sigma(b_1)x^2 + \cdots) + (a_2 x + a_3 x^2 + \cdots)(\sigma(b_0)x + \sigma(b_1)x^2 + \cdots).$$

Since $a_1 \sigma(b_j) = 0$ for all $j$, then

$$0 = (a_2 x + a_3 x^2 + \cdots)(\sigma(b_0)x + \sigma(b_1)x^2 + \cdots)$$
Let $a, i, j, k$ be an $\sigma$-(sps) Armendariz ring. Then for $f = \sum_{i=0}^{\infty} a_i x^i$, $g = \sum_{j=0}^{\infty} b_j x^j$ and $h = \sum_{k=0}^{\infty} c_k x^k \in R[[x; \sigma]]$, if $fgh = 0$ then $a_i b_j c_k = 0$ for all $i, j, k$.

Proof. Note that, if $fg = 0$ then $a_i g = 0$ for all $i$. Suppose that $fgh = 0$ then $a_i (gh) = 0$ for all $i$, and so $(a_i g) h = 0$ for all $i$. Therefore $a_i b_j c_k = 0$ for all $i, j, k$.

**Proposition 2.5.** Let $R$ be an $\sigma$-(sps) Armendariz ring. Then

1. $R$ is reversible if and only if $R[[x; \sigma]]$ is reversible.
2. $R$ is symmetric if and only if $R[[x; \sigma]]$ is symmetric.

Proof. If $R[[x; \sigma]]$ is symmetric (reversible) then $R$ is symmetric (reversible). Conversely, (1). Let $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \sigma]]$, if $fg = 0$ then $a_i b_j = 0$ for all $i$ and $j$. By [4, Theorem 3.3 (3ii)], we have $\sigma^j(a_i)b_j = 0$ for all $i$ and $j$. Since $R$ is reversible, we obtain $b_j \sigma^j(a_i) = 0$ for all $i$ and $j$. Thus $gf = \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} b_j \sigma^j(a_i)x^i = 0$. (2). We will use freely [4, Theorem 3.3 (3ii)], reversibility and symmetry of $R$. Let $f = \sum_{i=0}^{\infty} a_i x^i$, $g = \sum_{j=0}^{\infty} b_j x^j$ and $h = \sum_{k=0}^{\infty} c_k x^k \in R[[x; \sigma]]$, if $fgh = 0$ then $a_i b_j c_k = 0$ for all $i, j$ and $k$, by Lemma 2.4. Then for all $i, j, k$ we have $b_j c_k a_i = 0 \Rightarrow \sigma^k(b_j)c_k a_i = 0 \Rightarrow a_i \sigma^k(b_j)c_k = 0 \Rightarrow a_i c_k \sigma^k(b_j) = 0 \Rightarrow c_k \sigma^k(b_j)a_i = 0 \Rightarrow a_i \sigma^i[\sigma^k(b_j)] a_i = 0 \Rightarrow a_i \sigma^i[c_k \sigma^k(b_j)] = 0$. Thus $fgh = 0$.

The next Lemma gives a relationship between $\sigma$-reversibility ($\sigma$-symmetry) and reversibility (symmetry).

**Lemma 2.6 ([5, Lemma 3.1]).** Let $R$ be a ring and $\sigma$ an endomorphism of $R$. If $R$ satisfies the condition (C$_\sigma$). Then

1. $R$ is reversible if and only if $R$ is $\sigma$-reversible;
2. $R$ is symmetric if and only if $R$ is $\sigma$-symmetric.

**Theorem 2.7.** Let $R$ be an $\sigma$-(sps) Armendariz ring. The following statements are equivalent:

1. $R$ is reversible (symmetric);
2. $R$ is $\sigma$-reversible ($\sigma$-symmetric);
3. $R$ is right $\sigma$-reversible (right $\sigma$-symmetric);
4. $R[[x; \sigma]]$ is reversible (symmetric).
Proof. We prove the reversible case (the same for the symmetric case).

(1) $\iff$ (4). By Proposition 2.5.

(1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Immediately from Lemma 2.6.

(3) $\Rightarrow$ (1). Let $a,b \in R$, if $ab = 0$ then $b\sigma(a) = 0$ (right $\sigma$-reversibility), so $ba = 0$ (condition $(C_\sigma)$).

\[ \] 

3 Related Topics

In this section we turn our attention to the relationship between the Baerness, quasi-Baerness and p.p.-property of a ring $R$ and these of the skew power series ring $R[[x;\sigma]]$ in case $R$ is right $\sigma$-reversible. For a nonempty subset $X$ of $R$, we write $r_R(X) = \{c \in R|dc = 0 \text{ for any } d \in X\}$ which is called the right annihilator of $X$ in $R$.

**Lemma 3.1.** If $R$ is a right $\sigma$-reversible ring with $\sigma(1) = 1$. Then

1. $\sigma(e) = e$ for all idempotent $e \in R$;
2. $R$ is abelian.

**Proof.** (1) Let $e$ an idempotent of $R$. We have $e(1-e) = (1-e)e = 0$ then $(1-e)\sigma(e) = e\sigma((1-e)) = 0$, so $\sigma(e) - e\sigma(e) = e - e\sigma(e) = 0$, therefore $\sigma(e) = e$. (2) Let $r \in R$ and $e$ an idempotent of $R$. We have $e(1-e) = 0$ then $e(1-e)r = 0$, since $R$ is right $\sigma$-reversible then $(1-e)r\sigma(e) = 0 = (1-e)re = 0$, so $re = ere$. Since $(1-e)e = 0$, we have also $er = ere$. Then $R$ is abelian. \[ \]

**Lemma 3.2.** Let $R$ be a right $\sigma$-reversible ring with $\sigma(1) = 1$, then the set of all idempotents in $R[[x;\sigma]]$ coincides with the set of all idempotents of $R$. In this case $R[[x;\sigma]]$ is abelian.

**Proof.** We adapt the proof of [3, Theorem 2.13(iii)] for $R[[x;\sigma]]$. Let $f^2 = f \in R[[x;\sigma]]$, where $f = f_0 + f_1x + f_2x^2 + \cdots$. Then

\[ \sum_{\ell=0}^{\infty} \sum_{i+j=\ell} f_i \sigma^i(f_j)x^\ell = \sum_{\ell=0}^{\infty} f_\ell x^\ell. \]

For $\ell = 0$, we have $f_0^2 = f_0$. For $\ell = 1$, we have $f_0f_1 + f_1\sigma(f_0) = f_1$, but $f_0$ is central and $\sigma(f_0) = f_0$, so $f_0f_1 + f_1f_0 = f_1$, a multiplication by $(1-f_0)$ on the left hand gives $f_1 = f_0f_1$, and so $f_1 = 0$. For $\ell = 2$, we have $f_0f_2 + f_1\sigma(f_1) + f_2\sigma^2(f_0) = f_2$, so $f_0f_2 + f_2f_0 = f_2$ (because $f_1 = 0$ and $\sigma^2(f_0) = f_0$), a multiplication by $(1-f_0)$ on the left hand gives $f_0f_2 = f_2 = 0$. Continuing this procedure yields $f_i = 0$ for all $i \geq 1$. Consequently, $f = f_0 = f_0^2 \in R$. Since $R$ is abelian then $R[[x;\sigma]]$ is abelian. \[ \]
Kaplansky [13], introduced the concept of \textit{Baer rings} as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [7], a ring \( R \) is called \textit{quasi-Baer} if the right annihilator of each right ideal of \( R \) is generated (as a right ideal) by an idempotent. It is well-known that these two concepts are left-right symmetric. A ring \( R \) is called a \textit{right (left) p.p.-ring} if the right (left) annihilator of an element of \( R \) is generated by an idempotent. \( R \) is called a \textit{p.p.-ring} if it is both a right and left p.p.-ring.

**Theorem 3.3.** Let \( R \) be a right \( \sigma \)-reversible ring with \( \sigma(1) = 1 \). Then

1. \( R \) is a Baer ring if and only if \( R[[x; \sigma]] \) is a Baer ring;
2. \( R \) is a quasi-Baer ring if and only if \( R[[x; \sigma]] \) is a quasi-Baer ring.

**Proof.** (\( \Rightarrow \)). Suppose that \( R \) is Baer. Let \( A \) be a nonempty subset of \( R[[x; \sigma]] \) and \( A^\ast \) be the set of all coefficients of elements of \( A \). Then \( A^\ast \) is a nonempty subset of \( R \) and so \( r_R(A^\ast) = eR \) for some idempotent element \( e \in R \). Since \( e \in r_R([x; \sigma])A \) by Lemma 3.1. We have \( eR[[x; \sigma]] \subseteq r_R([x; \sigma])A \). Now, let \( 0 \neq q = b_0 + b_1x + b_2x^2 + \cdots \in r_R([x; \sigma])A \). Then \( Aq = 0 \) and hence \( pq = 0 \) for any \( p \in A \). Let \( p = a_0 + a_1x + a_2x^2 + \cdots, \) then

\[
pq = \sum_{\ell \geq 0} \sum_{i+j=\ell} a_i \sigma^i(b_j)x^\ell = 0.
\]

- \( \ell = 0 \) implies \( a_0b_0 = 0 \) then \( b_0 \in r_R(A^\ast) = eR \).
- \( \ell = 1 \) implies \( a_0b_1 + a_1 \sigma(b_0) = 0 \), since \( b_0 = eb_0 \) and \( \sigma(e) = e \) then \( a_0b_1 + a_1 \sigma(b_0) = 0 \), but \( a_1 = 0 \) so \( a_0b_1 = 0 \) and hence \( b_1 \in r_R(A^\ast) \).
- \( \ell = 2 \) implies \( a_0b_2 + a_1 \sigma(b_1) + a_2 \sigma^2(b_0) = 0 \), then \( a_0b_2 + a_1 \sigma(b_1) + a_2 \sigma^2(b_0) = 0 \), but \( a_1 \sigma(b_1) = a_2 \sigma^2(b_0) = 0 \), hence \( a_0b_2 = 0 \). Then \( b_2 \in r_R(A^\ast) \).

Continuing this procedure yields \( b_0, b_1, b_2, b_3, \cdots \in r_R(A^\ast) \). So, we can write \( q = eb_0 + eb_1x + eb_2x^2 + \cdots \in eR[[x; \sigma]] \). Therefore \( eR[[x; \sigma]] = r_R([x; \sigma])A \).

Consequently, \( R[[x; \sigma]] \) is a Baer ring.

Conversely, Suppose that \( R[[x; \sigma]] \) is Baer. Let \( B \) be a nonempty subset of \( R \). Then \( r_R([x; \sigma])(B) = eR[[x; \sigma]] \) for some idempotent \( e \in R \) by Lemma 3.2. Thus \( r_R(B) = r_R([x; \sigma])(B) \cap R = eR[[x; \sigma]] \cap R = eR \). Therefore \( R \) is Baer.

The proof for the case of the quasi-Baer property follows in a similar fashion; In fact, for any right ideal \( A \) of \( R[[x; \sigma]] \), take \( A^\ast \) as the right ideal generated by all coefficients of elements of \( A \).

From [10, Example 20], \( R = M_2(\mathbb{Z}) \) is a Baer ring and \( R[[x]] \) is not Baer. Clearly \( R \) is not reversible. So that, the “right \( \sigma \)-reversibility” condition in Theorem 3.3(1) is not superfluous.
According to Annin [2], a ring $R$ is $\sigma$-compatible if for each $a, b \in R$, $a\sigma(b) = 0$ if and only if $ab = 0$. Hashemi and Moussavi [9, Corollary 2.14] have proved Theorem 3.3(2), when $R$ is $\sigma$-compatible. Consider $R$ and $\sigma$ as in Example 2.2. Since $R$ is a domain then it is right $\sigma$-reversible (with $\sigma(1) = 1$). Also $R$ is not $\sigma$-compatible (so $R$ does not satisfy the condition $(C_\sigma)$), because $\sigma$ is not a monomorphism. Therefore Theorem 3.3(2) is not a consequence of [9, Corollary 2.14]. On other hand, if $R$ is reversible then $\sigma$-compatibility implies right $\sigma$-reversibility. But, if $R$ is not reversible, we can easily see that this implication does not hold.

**Theorem 3.4.** Let $R$ be a right $\sigma$-reversible ring with $\sigma(1) = 1$. If $R[[x; \sigma]]$ is a p.p.-ring then $R$ is a p.p.-ring.

**Proof.** Suppose that $R[[x; \sigma]]$ is a right p.p.-ring. Let $a \in R$, then there exists an idempotent $e \in R$ such that $r_{R[[x; \sigma]]}(a) = eR[[x; \sigma]]$ by Lemma 3.2. Hence $r_R(a) = eR$, and therefore $R$ is a right p.p.-ring.

Also, in Example 2.2, $R$ is not $\sigma$-(sps) Armendariz. So Theorem 3.3 and Theorem 3.4 are not consequences of [4, Theorem 3.2].

Since $\sigma$-rigid rings are right $\sigma$-reversible [15, Theorem 2.8 (1)], we have the following Corollaries.

**Corollary 3.5 ([10, Theorem 21]).** Let $R$ be an $\sigma$-rigid ring. Then $R$ is a Baer ring if and only if $R[[x; \sigma]]$ is a Baer ring.

**Corollary 3.6 ([10, Corollary 22]).** Let $R$ be an $\sigma$-rigid ring. Then $R$ is a quasi-Baer ring if and only if $R[[x; \sigma]]$ is a quasi-Baer ring.

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**References**


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